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The Logical Structure of Scientific Explanation and Prediction: Planetary Orbits in a Sun's Gravitational Field

Abstract. We present a logically detailed case-study of explanation and prediction in Newtonian mechanics. The case in question is that of a planet's elliptical orbit in the Sun's gravitational field. Care is taken to distinguish the respective contributions of the mathematics that is being applied, and of the empirical hypotheses that receive a mathematical formulation. This enables one to appreciate how in this case the overall logical structure of scientific explanation and prediction is exactly in accordance with the hypothetico-deductive model.

Keywords: Kepler, Newton, Laws of Motion, Law of Gravitation, planetary orbit, scientific explanation, hypothetico-deductive model, hypotheses, predictions, natural logicism, kinematics, conics, polar coordinates, derivatives, ellipse, focus, directrix, eccentricity, semi-latus rectum, equation of energy.

1. Introduction

The aim of this essay is twofold. First, we want to bring out clearly just what a paradigm example of the hypothetico-deductive model of explanation we have in the logico-mathematical deductions of **Kepler's three Laws of Planetary Motion** (the *explananda* in question) from **Newton's Laws of Motion and Law of Gravitation** plus an empirical assumption about the total energy of a planet (the *hypotheses* in question). Secondly, we want to set out these deductions with enough logical detail to make plausible the assurance that, had we but pages wide enough for the sideways sprawl of rigorous natural deductions *à la* Gentzen, the reasoning set out here could be fully formalized. We mean 'fully formalized' in utter seriousness—in a system of introduction and elimination rules for all the logico-mathematical operators involved. The author assures the reader that *all* the mathematics

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of derivatives appealed to here has been derived in that fashion.[1] Such derivations form part of a program that the author calls *natural logicism.* The natural logicist treats each synthetic branch of mathematics on its own primitive terms, formulating extended Gentzen-style introduction and elimination rules for the logico-mathematical operators involved. It works for arithmetic;[2] it works for both projective[3] and Euclidean geometry; and here we show the fruits of its working for the real differential calculus.

The philosophically or methodologically minded reader who is willing to take the ensuing applied mathematical details on trust—or for whom the taste of these fruits is perhaps too tart—may skip the technical details and pick up the closing thread of philosophical and methodological discussion in §3, after Kepler's Laws have been derived as promised.

2. The applied mathematics of orbital analysis

Consider a planet (or point-particle) of mass m, orbiting around the Sun, of mass M. (We can also call the Sun the *central body* for the purposes of this discussion.) The word 'orbit' carries with it the implication that the motion of the planet is periodic, along a closed curve. (We set aside the problem that the perihelion of an orbit might actually precess. The supposition, for our explanatory problem, is that there is no such precession.) M is many orders of magnitude greater than m. We can therefore treat the Sun as fixed, at the origin of a polar coordinate system, rather than as itself orbiting around the common center of mass of both bodies.[4] With that approximation, the problem is to characterize the orbit of the planet, in terms of the angle θ and the distance r between centers of mass. The question is: *What kind of curve is the orbit?*

Newtonian mechanics provides the answer to this question. It is our concern here to lay bare the logical structure of the reasoning behind the answer.[5] We assume that the only force acting on the planet is the gravita-

[1] See [7], in which the logical details are furnished. Our reasoning in this essay proceeds at a level of informality that should enable the non-logician to understand it, but in sufficient detail to convince the reader that its completely formal regimentation in natural deduction is perfectly feasible.

[2] See [4] and [6].

[3] See [5].

[4] For a more detailed discussion of this issue, see the author's unpublished paper [8].

[5] The discussion that follows is indebted to [3], pp. 153–166. This was the author's student text on Newtonian dynamics, and he has not yet found any better. In this study we re-order the material, supply missing details in the mathematical reasoning, and offer more

tional attraction of the Sun, directed towards the Sun's center of mass. Thus by **Newton's First Law of Motion**, there is no component of acceleration that would take the planet out of the plane defined by (i) the line joining the two centers of mass, and (ii) any instantaneous velocity vector of the planet.

Kepler's Laws of Planetary Motion, which he formulated upon painstaking analysis of observational data,[6] are as follows. They are stated on the approximating assumption that the Sun is taken as fixed at the origin of an inertial frame of reference.

1. Any planet orbits the Sun in an ellipse, with the Sun at one focus.[7]
2. The radius from (the center of mass of) the Sun to (the center of mass of) any planet sweeps out equal areas in equal periods of time.
3. The ratio of the square of the planetary year to the cube of the semi-major axis[8] of its orbit is constant (the same for all planets).

The laws state regularities that are found in the observational data. Although going strictly beyond the data, **Kepler's First Law** does not provide the kind of *theoretical* answer that we seek to our question *What kind of curve is the orbit?*. By looking at the night sky, as it were, Kepler was able to say 'an ellipse'. But the question that we really want answered is: *Why* is it an ellipse? We seek a deeper explanation for this 'lower-level' regularity that Kepler found.

We have already appealed to **Newton's First Law of Motion**:

> Every body perseveres in its state of being at rest or of moving uniformly straight forward, except insofar as it is compelled to change its state by forces impressed,[9]

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by way of explanation. This is all in order to help the student of logic, methodology and philosophy of science to understand better the logico-mathematical structure of scientific explanation and empirical testing.

[6] His methods were both painstaking and pain-inflicting. A strong case has been made for the charge that Kepler poisoned his employer Tycho Brahe with mercury, in order to purloin the leatherbound volumes containing Brahe's records of astronomical observations. See [1] for details.

[7] An ellipse has two foci. In the special case of a circle, these coincide at the center. An ellipse can be characterized by reference to its two foci as the locus of a point the sum of whose distances from the foci is constant. In the treatment offered here, however, we shall use an alternative (but of course equivalent) definition of an ellipse, in terms of a focus and a line called the *directrix*. Details will follow.

[8] This notion is defined in below.

[9] See [2], at p. 416.

in setting up a two-dimensional, or planar, representation of the motion to be analyzed. **Newton's Law of Gravitation** implies that the gravitational force exerted by the Sun at time t on the planet is

$$\frac{G \cdot M \cdot m}{[d(t)]^2},$$

where G is the gravitational constant and $d(t)$ is the distance, at time t, between the centers of mass of the two bodies. **Newton's Second Law of Motion** implies that

$$F(t) = m \cdot a(t)$$

where $F(t)$ is the total force exerted on the planet at time t and $a(t)$ is the magnitude of its acceleration at time t in the direction in which the force is applied. Thus

$$\frac{G \cdot M \cdot m}{d(t)^2} = m \cdot a(t), \qquad \text{whence} \qquad \frac{G \cdot M}{d(t)^2} = a(t).$$

Note that this is the magnitude of an orbiting planet's *centripetal acceleration* that results from the Sun's gravitation. It would be the same for any planet in that orbit, regardless of its mass.

2.1. Some basics of the calculus

We turn now to some of the mathematics that we need. The branch of mathematics known as the calculus (the theory of *differentiation* and *integration* of functions of real numbers) supplies the mathematical theorems that find application in the Newtonian solution of our problem. (Indeed, Newton invented it for this purpose.) The *Fundamental Theorem on Integration* says that if two functions $f(x)$ and $g(x)$ have the same derivative with respect to x, then they differ by a constant:

$$\left[\forall x \left(\frac{d}{dx}f(x) = \frac{d}{dx}g(x)\right)\right] \;\rightarrow\; \exists a \forall x\, [f(x) = g(x) + a].$$

We have the simple theorems (for a any constant)

$$\frac{d}{dx}(a) = 0; \qquad \frac{d}{dx}(a \cdot x) = a; \qquad \frac{d}{dx}(a \cdot x^2) = 2 \cdot a \cdot x;$$

the additivity theorem

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x));$$

the theorem on differentiation of products:

$$\frac{d}{dx}(f(x) \cdot g(x)) = \frac{d}{dx}(f(x)) \cdot g(x) + f(x) \cdot \frac{d}{dx}(g(x));$$

and the *chain rule* for differentiating compositions of functions with respect to the innermost independent variable (here, x):

$$\frac{d}{dx}(f(g(x))) = \frac{d}{dg}(f(g)) \cdot \frac{d}{dx}(g(x)),$$

or, put more simply,

$$\frac{d}{dx}(f(g(x))) = \frac{df}{dg} \cdot \frac{dg}{dx}.$$

Two important derivatives are those of the so-called *harmonic* functions:

$$\frac{d}{dx}(\sin(x)) = \cos(x); \qquad \frac{d}{dx}(\cos(x)) = -\sin(x).$$

Hence also

$$\frac{d^2}{dx^2}(\sin(x)) = -\sin(x); \qquad \frac{d^2}{dx^2}(\cos(x)) = -\cos(x).$$

Thus both $\sin(x)$ and $\cos(x)$ are solutions to the second-order differential equation

$$\frac{d^2}{dt^2}(f(x)) = -f(x).$$

More generally, suppose $f(x)$ is of the form $a \cdot \cos(x+b)+c$, where a, b and c are constants. Then

$$\frac{d}{dx}(f(x)) = -a \cdot \sin(x+b) \qquad \text{and} \qquad \frac{d^2}{dx^2}(f(x)) = -a \cdot \cos(x+b),$$

whence

$$\frac{d^2}{dx^2}(f(x)) + f(x) = c.$$

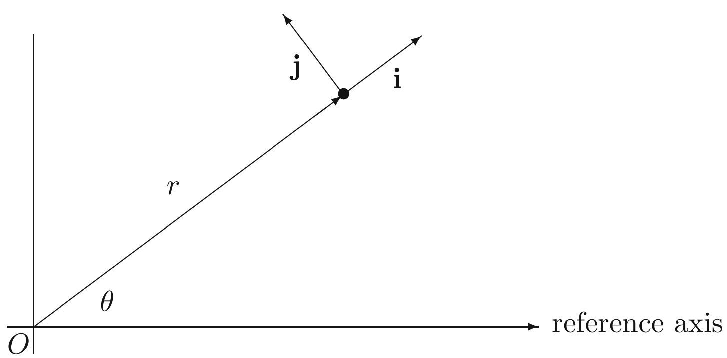
Thus this last second-order differential equation has as its general solution for $f(x)$ a function of the form $a \cdot \cos(x+b)+c$. One could also have 'sin' in place of 'cos' here. But, since $\cos(x) = \sin(x+\frac{\pi}{2})$, there is no loss of generality in dealing only with 'cos'.

2.2. Kinematics: position, velocity, acceleration

We turn now to an explanation of how *polar coordinates* are used in plane geometry. They are particularly apt for the kinematic description of motions of point-particles that move under the influence of a central force.

Kinematics is that branch of applied (or, perhaps more accurately: *applicable*) mathematics that furnishes the means for talking about the position, velocity and acceleration of a point-particle over time. Time is treated as a variable over the real numbers. The *spatial coordinates* that are used in order to locate the point in question are also real numbers. These spatial coordinates are time-dependent, so they are expressed as real-valued functions of time. These functions are continuous. (*Natura non fecit salta!*) In the applications to be considered here, the point-particles are moreover not acted on by any impulsive forces, such as would be exerted instantaneously in head-on collisions of inelastic bodies. But the bodies we are considering do not collide. Thus it may be assumed that the spatial-coordinate functions of time enjoy time-derivatives of arbitrary order. Certainly, their first and second time-derivatives exist.

A polar coordinate system for a plane involves fixing the *origin* O (a point lying in the plane) and the *reference axis*, a line (in the plane) through O:



Distance along the radius that joins the origin O to the point-particle (represented by the black dot) is represented by $r(t)$. When we write r, its dependence on the temporal parameter t is implicit.

The **position vector** is $r \cdot \mathbf{i}$, where $\mathbf{i}$ is the unit vector pointing from the point-particle along the radius away from O. Since $\mathbf{i}$ is a vector, it is identified only via the angle θ; $\mathbf{i}$ does not depend on the length r, since a vector is determined only by its length and its orientation (i.e. the direction in which it points). Altering r alters neither the length of $\mathbf{i}$—since $\mathbf{i}$ is, after all, a *unit* vector—nor its orientation. Thus $\mathbf{i}$ should really be thought of as $\mathbf{i}(\theta)$. Perpendicular to $\mathbf{i}$ is the unit vector $\mathbf{j}$, also pointing from the point-particle, and in a counterclockwise direction. For similar reasons as just laid out, $\mathbf{j}$ too should really be thought of as $\mathbf{j}(\theta)$.

The following claims are intuitively obvious, and important for what follows:

$$\frac{d\mathbf{i}}{d\theta} = \mathbf{j} \qquad \frac{d\mathbf{j}}{d\theta} = -\mathbf{i}.$$

Just as the unit vectors $\mathbf{i}$ and $\mathbf{j}$ depend on θ, so too do they depend on time t. Thus one could also think of them as $\mathbf{i}(t)$ and $\mathbf{j}(t)$. But this requires that at time t the *angular coordinate* θ of the point-particle be determined. For both $\mathbf{i}$ and $\mathbf{j}$ point in their respective directions *from the point-particle* whose velocity and acceleration they help to describe, and these directions depend on θ.

The point-particle's (time-dependent) **velocity vector** $\mathbf{v}(t)$ is the rate of change (at time t) of its position vector—that is,

$$\mathbf{v}(t) = \frac{d}{dt}(r \cdot \mathbf{i}).$$

Bear in mind that both factors of the product here being differentiated depend on t. Thus by the law for differentiating products,

$$\mathbf{v}(t) = \dot{r} \cdot \mathbf{i} + r \cdot \frac{d\mathbf{i}}{dt}.$$

By the chain rule we have

$$\frac{d\mathbf{i}}{dt} = \frac{d\mathbf{i}}{d\theta} \cdot \frac{d\theta}{dt}.$$

But

$$\frac{d\mathbf{i}}{d\theta} = \mathbf{j}.$$

Hence

$$\mathbf{v}(t) = \dot{r} \cdot \mathbf{i} + r \cdot \dot{\theta} \cdot \mathbf{j}.$$

The point-particle's (time-dependent) **acceleration vector** $\mathbf{a}(t)$ is the rate of change (at time t) of its velocity vector—that is,

$$\mathbf{a}(t) = \frac{d}{dt}(\mathbf{v}(t)).$$

Substituting for $\mathbf{v}(t)$, we obtain

$$\mathbf{a}(t) = \frac{d}{dt}(\dot{r} \cdot \mathbf{i} + r \cdot \dot{\theta} \cdot \mathbf{j}).$$

The derivative of a sum of two functions is the sum of the derivatives of those functions; hence

$$\mathbf{a}(t) = \frac{d}{dt}(\dot{r} \cdot \mathbf{i}) + \frac{d}{dt}(r \cdot \dot{\theta} \cdot \mathbf{j}).$$

We shall work out the left summand $\frac{d}{dt}(\dot{r} \cdot \mathbf{i})$ first, then work out the second summand $\frac{d}{dt}(r \cdot \dot{\theta} \cdot \mathbf{j})$. We have

$$\frac{d}{dt}(\dot{r} \cdot \mathbf{i}) = \ddot{r} \cdot \mathbf{i} + \dot{r} \cdot \frac{d\mathbf{i}}{dt}.$$

By the chain rule again we have

$$\frac{d}{dt}(\dot{r} \cdot \mathbf{i}) = \ddot{r} \cdot \mathbf{i} + \dot{r} \cdot \frac{d\mathbf{i}}{d\theta} \cdot \frac{d\theta}{dt}.$$

Hence

$$\frac{d}{dt}(\dot{r} \cdot \mathbf{i}) = \ddot{r} \cdot \mathbf{i} + \dot{r} \cdot \mathbf{j} \cdot \dot{\theta}.$$

Now we turn to the second summand, which we shall express as $\frac{d}{dt}([r \cdot \dot{\theta}] \cdot \mathbf{j})$ so that we can apply the rule for differentiating products. We have

$$\begin{aligned}\frac{d}{dt}([r \cdot \dot{\theta}] \cdot \mathbf{j}) &= \left(\frac{d}{dt}[r \cdot \dot{\theta}]\right) \cdot \mathbf{j} \ + \ [r \cdot \dot{\theta}] \cdot \frac{d\mathbf{j}}{dt} \\ &= (r \cdot \ddot{\theta} + \dot{\theta} \cdot \dot{r}) \cdot \mathbf{j} \ + \ [r \cdot \dot{\theta}] \cdot \frac{d\mathbf{j}}{dt}.\end{aligned}$$

Now note

$$\frac{d\mathbf{j}}{dt} = \frac{d\mathbf{j}}{d\theta} \cdot \frac{d\theta}{dt} = -\mathbf{i} \cdot \dot{\theta}.$$

Hence for the second summand we have

$$\frac{d}{dt}([r \cdot \dot{\theta}] \cdot \mathbf{j}) = (r \cdot \ddot{\theta} + \dot{\theta} \cdot \dot{r}) \cdot \mathbf{j} \ - \ [r \cdot \dot{\theta}] \cdot \mathbf{i} \cdot \dot{\theta}.$$

Having worked out the two summands for acceleration, we can add them together to obtain

$$\mathbf{a}(t) = (\ddot{r} \cdot \mathbf{i} + \dot{r} \cdot \mathbf{j} \cdot \dot{\theta}) \ + \ ((r \cdot \ddot{\theta} + \dot{\theta} \cdot \dot{r}) \cdot \mathbf{j} \ - \ [r \cdot \dot{\theta}] \cdot \mathbf{i} \cdot \dot{\theta}).$$

Regrouping terms, we conclude

$$\mathbf{a}(t) = (\ddot{r} - r \cdot \dot{\theta}^2) \cdot \mathbf{i} \ + \ (2 \cdot \dot{r} \cdot \dot{\theta} + r \cdot \ddot{\theta}) \cdot \mathbf{j}.$$

Note that

$$\begin{aligned}\frac{d}{dt}(r^2 \cdot \dot{\theta}) &= \frac{d}{dt}(r^2) \cdot \dot{\theta} + r^2 \cdot \frac{d}{dt}(\dot{\theta}) \\ &= 2 \cdot r \cdot \dot{r} \cdot \dot{\theta} + r^2 \cdot \ddot{\theta} \\ &= r \cdot (2 \cdot \dot{r} \cdot \dot{\theta} + r \cdot \ddot{\theta})\end{aligned}$$

whence

$$\frac{1}{r} \cdot \frac{d}{dt}(r^2 \cdot \dot{\theta}) = 2 \cdot \dot{r} \cdot \dot{\theta} + r \cdot \ddot{\theta}.$$

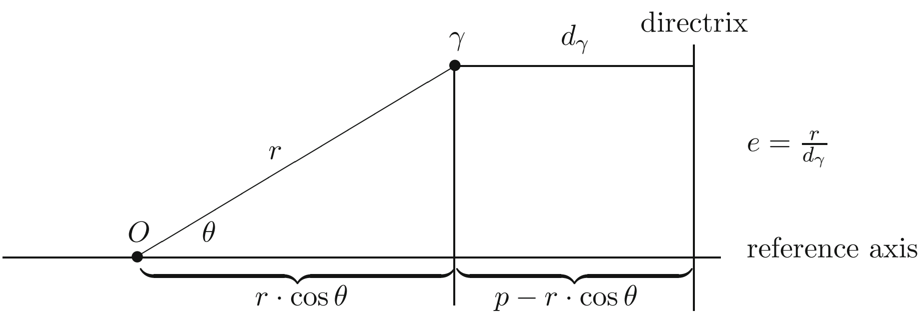
Thus we can also write

$$\mathbf{a}(t) = (\ddot{r} - r \cdot \dot{\theta}^2) \cdot \mathbf{i} \ + \ \left(\frac{1}{r} \cdot \frac{d}{dt}(r^2 \cdot \dot{\theta})\right) \cdot \mathbf{j}.$$

We now have the point-particle's acceleration vector expressed in terms of its (radial) **i**-component and its (angular) **j**-component.

2.3. Geometry

So much for kinematics. Now for some geometry. (Only in §2.4 do we *apply* any of the kinematics or geometry.) A celebrated definition of the plane figures called **conics**—ellipse, parabola or hyperbola—involves a fixed point O, called the *focus*, and a fixed line (in the plane) not containing O, called the *directrix*. The perpendicular distance from the focus O to the directrix is of course constant, and is called p. The line containing O and perpendicular to the directrix is called the *reference axis*. Consider now a point γ with polar coordinates (r, θ) with respect to O and the reference axis. The locus of such points γ is called a *conic* just in case the ratio of the distance r (from γ to O) to the perpendicular distance d_γ from γ to the directrix is a constant (call it e):



As is evident from the diagram, $d_\gamma = p - r \cdot \cos\theta$. Thus the defining condition for the locus of points γ to be a conic is that

$$\frac{r}{p - r \cdot \cos\theta} = e$$

The exact kind of conic depends on the value of e:

$e = 0$	circle
$0 < e < 1$	ellipse
$e = 1$	parabola
$1 < e$	hyperbola

When θ is a right angle, the distance of γ from O is called the *semi-latus rectum* of the conic, and will be denoted as λ. The point γ is then the distance p away from the directrix. Thus we have

$$e = \frac{\lambda}{p}, \quad \text{or} \quad \lambda = e \cdot p.$$

From the defining equation given above for a conic one can reason as follows:

$$r = e \cdot p - e \cdot r \cdot \cos\theta$$

$$r + e \cdot r \cdot \cos\theta = e \cdot p$$

$$r \cdot (1 + e \cdot \cos\theta) = e \cdot p$$

$$r \cdot (1 + e \cdot \cos\theta) = \lambda.$$

Since $\cos\theta$ is harmonic, it would not matter if one added a constant b to θ (corresponding to a rotation of b radians, about O, of the reference line for the measurement of angular displacement). So we could write

$$\boxed{r \cdot (1 + e \cdot \cos(\theta + b)) = \lambda}$$

The boxed equation defines a conic, in polar coordinates, in terms of its eccentricity e and its semi-latus rectum λ, in nicely general terms. For it does not require the reference axis for the purpose of defining a conic to coincide with the reference line from which angular displacement is measured when working with polar coordinates. In practice, however, it can be convenient to let the reference line coincide with the reference axis (whence $b = 0$).

Definition 1. $u =_{df} \frac{1}{r}$

It follows from Definition 1 that the last boxed equation, for the definition of a conic in terms of its eccentricity e and semi-latus rectum λ, could be written

$$\boxed{u = \frac{1}{\lambda} \cdot (1 + e \cdot \cos(\theta + b))}$$

If the eccentricity e lies between 0 and 1, then the conic is an ellipse. If we inspect the equation in the form

$$r = \frac{\lambda}{1 + e \cdot \cos(\theta + b)}$$

we see that the values of r are constrained to lie between a minimum of $\frac{\lambda}{1+e}$ and a maximum of $\frac{\lambda}{1-e}$ (inclusive). This is because the values of the cosine function lie between -1 and 1 (inclusive). The minimum value $\frac{\lambda}{1+e}$ is attained when the input to the cosine function is 0 (for $\cos(0) = 1$), and the maximum value $\frac{\lambda}{1-e}$ is attained when that input is π (for $\cos(\pi) = -1$). For both these input values, the point tracing the ellipse lies on the reference axis. So the ellipse cuts the reference axis twice. The segment of the reference axis between these two points is called the *major axis* of the ellipse. We denote half of that length by α, and call it the *semi-major axis*.

When the point γ tracing the ellipse is on the reference axis, between the focus O and the directrix, its distance x from O and its distance $(p - x)$ from the directrix are in the same ratio as λ is to p. That is,

$$\frac{x}{p - x} = \frac{\lambda}{p}.$$

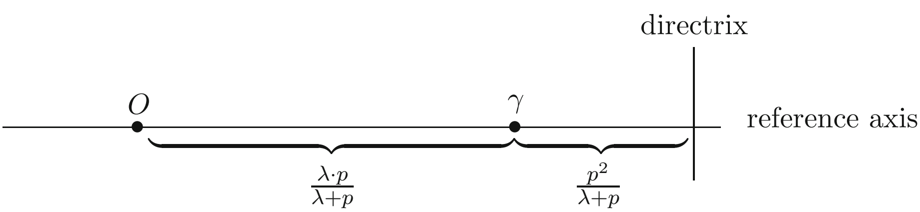
From this it follows that $x \cdot p = \lambda \cdot p - \lambda \cdot x$, whence $x \cdot (\lambda + p) = \lambda \cdot p$ and

$$x = \frac{\lambda \cdot p}{\lambda + p}.$$

Another consequence is that

$$p - x = \frac{p^2}{\lambda + p}.$$

Our diagram can therefore be labelled as follows:



When r attains its maximum value, with the point λ that traces the ellipse lying on the reference axis to the left of O, its distance from O is $2 \cdot \alpha - \frac{\lambda \cdot p}{\lambda + p}$; and its distance from the directrix is $2 \cdot \alpha + \frac{p^2}{\lambda + p}$. Thus we have

$$\frac{2 \cdot \alpha - \frac{\lambda \cdot p}{\lambda + p}}{2 \cdot \alpha + \frac{p^2}{\lambda + p}} = \frac{\lambda}{p}.$$

From this the following mushroom cloud of equations can be inferred, *seriatim*:

$$p \cdot (2 \cdot \alpha - \frac{\lambda \cdot p}{\lambda + p}) = \lambda \cdot (2 \cdot \alpha + \frac{p^2}{\lambda + p})$$

$$\frac{p \cdot (2 \cdot \alpha \cdot (\lambda + p) - \lambda \cdot p)}{\lambda + p} = \frac{\lambda \cdot (2 \cdot \alpha \cdot (\lambda + p) + p^2)}{\lambda + p}$$

$$p \cdot (2 \cdot \alpha \cdot (\lambda + p) - \lambda \cdot p) = \lambda \cdot (2 \cdot \alpha \cdot (\lambda + p) + p^2)$$

$$p \cdot (2 \cdot \alpha \cdot \lambda + 2 \cdot \alpha \cdot p - \lambda \cdot p) = \lambda \cdot (2 \cdot \alpha \cdot \lambda + 2 \cdot \alpha \cdot p + p^2)$$

$$2 \cdot \alpha \cdot \lambda \cdot p + 2 \cdot \alpha \cdot p^2 - \lambda \cdot p^2 = 2 \cdot \alpha \cdot \lambda^2 + 2 \cdot \alpha \cdot \lambda \cdot p + \lambda \cdot p^2$$

$$2 \cdot \alpha \cdot p^2 - \lambda \cdot p^2 = 2 \cdot \alpha \cdot \lambda^2 + \lambda \cdot p^2$$

$$2 \cdot \alpha \cdot p^2 = 2 \cdot \alpha \cdot \lambda^2 + 2 \cdot \lambda \cdot p^2$$

$$\alpha \cdot p^2 = \alpha \cdot \lambda^2 + \lambda \cdot p^2$$

$$\lambda \cdot p^2 = \alpha \cdot p^2 - \alpha \cdot \lambda^2$$

$$\lambda = \frac{\alpha \cdot p^2 - \alpha \cdot \lambda^2}{p^2}$$

$$\lambda = \alpha \cdot \left(1 - \frac{\lambda^2}{p^2}\right)$$

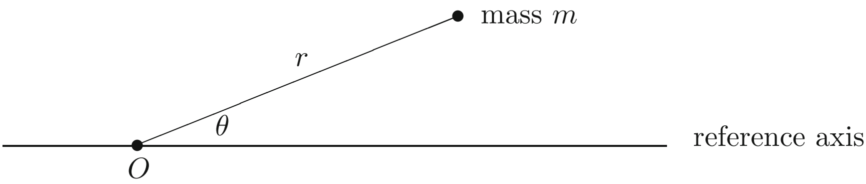
$$\lambda = \alpha \cdot \left(1 - \left(\frac{\lambda}{p}\right)^2\right)$$

$$\lambda = \alpha \cdot (1 - e^2).$$

We shall make use of this last equation in due course. Note that so far we have been doing only mathematics, making no empirical assumptions.

2.4. Motion under a central force

We are now in a position to address motion under a central force, and apply the kinematics and the geometry of conics developed above.



Definition 2. *The angular momentum of the mass m about the origin O is $(m \cdot r \cdot (\mathbf{j}$-component of velocity$))$, i.e. $m \cdot r^2 \cdot \dot{\theta}$.*

2.4.1. Kepler's Second Law

If the only force acting on m is always along the radius joining m to O (no matter whether centrifugal or centripetal), then the angular momentum of m about O is a constant—whence $r^2 \cdot \dot{\theta}$, the *angular momentum per unit mass*, is a constant ($= h$, say). Thus the *areal velocity* $\frac{1}{2} \cdot r^2 \cdot \dot{\theta}$ is constant. This is **Kepler's Second Law of Planetary Motion** (see above).

It is a little-appreciated fact that **Kepler's Second Law** holds even if the central force is centrifugal, i.e. repulsive. All that is important is that the force be radial, so that it has no angular component. To the extent that **Kepler's Second Law** reports a regularity in astronomical observations, it is explained merely by postulating that a planet moves under the influence of a central force alone. It does not matter whether the force in question is attractive or repulsive; nor does it matter how its magnitude might vary as a function of the planet's position in relation to the Sun.

2.4.2. Equations of motion under a central force

It follows from Definition 1 that for motion under a central force $\dot{\theta} = h \cdot u^2$ and $h = \dot{\theta}/u^2$. So the first and second time-derivatives of r are

$$\dot{r} \;=\; \frac{dr}{du} \cdot \frac{du}{d\theta} \cdot \frac{d\theta}{dt} \;=\; \frac{d}{du}(u^{-1}) \cdot \frac{du}{d\theta} \cdot \dot{\theta} \;=\; \left(-\frac{1}{u^2}\right) \cdot \frac{du}{d\theta} \cdot \dot{\theta} \;=\; -h \cdot \frac{du}{d\theta}$$

$$\ddot{r} = \frac{d\dot{r}}{d\theta} \cdot \frac{d\theta}{dt} = \frac{d}{d\theta}\left(-h \cdot \frac{du}{d\theta}\right) \cdot \dot{\theta} = -h \cdot \frac{d^2u}{d\theta^2} \cdot \dot{\theta} = -h \cdot \frac{d^2u}{d\theta^2} \cdot h \cdot u^2 = -h^2 \cdot u^2 \cdot \frac{d^2u}{d\theta^2}.$$

Now we are able to express the **i**-coordinate of the acceleration of the mass m (under a central force) in terms of u and θ:

$$\ddot{r}-r\cdot\dot{\theta}^2 \;=\; -h^2\cdot u^2\cdot\frac{d^2u}{d\theta^2}-r\cdot(h\cdot u^2)^2 \;=\; -h^2\cdot u^2\cdot\frac{d^2u}{d\theta^2}-h^2\cdot u^3 \qquad (\text{since } r=\tfrac{1}{u})$$

$$= -h^2\cdot u^2\cdot\left(\frac{d^2u}{d\theta^2}+u\right).$$

Another useful quantity is the *square of the velocity* of the mass m under a central force:

$$\dot{r}^2+r^2\cdot\dot{\theta}^2 \;=\; h^2\cdot\left(\frac{du}{d\theta}\right)^2+r^2\cdot(h\cdot u^2)^2 \;=\; h^2\cdot\left[\left(\frac{du}{d\theta}\right)^2+u^2\right].$$

We now have the following equations for motion under a central force:

$$\boxed{\dot{r}=-h\cdot\tfrac{du}{d\theta}} \quad \boxed{\ddot{r}=-h^2\cdot u^2\cdot\tfrac{d^2u}{d\theta^2}}$$

$$\boxed{\ddot{r}-r\cdot\dot{\theta}^2=-h^2\cdot u^2\cdot\left(\tfrac{d^2u}{d\theta^2}+u\right)} \quad \boxed{\dot{r}^2+r^2\cdot\dot{\theta}^2=h^2\cdot\left[\left(\tfrac{du}{d\theta}\right)^2+u^2\right]}$$

By **Newton's Second Law of Motion**, we have $m\mathbf{a}= m\mathbf{F}$, where $\mathbf{F}$ is the central force per unit mass (equivalently, the radial component of acceleration, namely $\ddot{r}-r\cdot\dot{\theta}^2$, in the direction of the radial unit vector $\mathbf{i}$). Now, the radial unit vector points outwards from O. Let F_O be the magnitude of the *inward* component of $\mathbf{F}$, so that $\mathbf{F}= (-F_O)\cdot\mathbf{i}$. Then we have

$$\ddot{r}-r\cdot\dot{\theta}^2=-F_O=-h^2\cdot u^2\cdot\left(\frac{d^2u}{d\theta^2}+u\right).$$

A little manipulation of terms yields the following equation of motion for a mass m under the influence of a central force of magnitude F_O:

$$\boxed{\tfrac{d^2u}{d\theta^2}+u=\tfrac{F_O}{h^2\cdot u^2}}$$

We now need to solve for u $(=\frac{1}{r})$ as a function of θ. Note that u occurs in the denominator on the right-hand side.

2.4.3. Enter gravitation: motion under a central, attractive, inverse-square force

We have not yet made any assumption about how the *magnitude* F_O of the central force varies as a function of the position of the mass m in relation to O (the point through which the central force always acts). Let us now adopt as a further assumption

NEWTON'S INVERSE SQUARE LAW OF GRAVITATION: $F_O = \dfrac{\mu}{r^2}$

where μ $(= G \cdot M)$ is a constant. We can also express this as

$$F_O = \mu \cdot u^2.$$

Then our equation of motion for a mass m under the influence of a central *gravitational* force, which is attractive, and which varies in proportion to the inverse square of the distance r, becomes

$$\boxed{\tfrac{d^2u}{d\theta^2} + u = \tfrac{\mu}{h^2}}$$

where the right-hand side is now a *constant*.

We saw earlier that the general form of solution to a second-order differential equation of this form is

$$u(\theta) = a \cdot \cos(\theta + b) + \frac{\mu}{h^2}.$$

Recall that $u = \frac{1}{r}$. Thus

$$1 = r \cdot \left(a \cdot \cos(\theta + b) + \frac{\mu}{h^2}\right)$$

whence

$$r \cdot \left(1 + \frac{a \cdot h^2}{\mu} \cdot \cos(\theta + b)\right) = \frac{h^2}{\mu}.$$

Compare this with our earlier equation

$$\boxed{r \cdot (1 + e \cdot \cos(\theta + b)) = \lambda}$$

It is clear now that motion under a central, *attractive*, inverse-square force describes a conic. So we have now found the answer to our question.[10]

[10] The author notes with satisfaction that, despite the appeal made to the so-called fundamental theorem of *integration*, not a single integral sign has appeared in the foregoing

2.4.4. Kepler's First Law

Kepler's First Law is more specific than the result derived above, about the paths of point-masses under gravitational attraction being conics. We need some further explanation of why it is that planetary orbits are *ellipses*, and not parabolas or hyperbolas. It is not enough to say that, by definition as it were, planets are the kinds of bodies that orbit repeatedly around the sun, and that therefore only an elliptical orbit is open to a planet. The deeper scientific question is: what is it about a heavenly body *that makes it behave this way* in relation to the Sun, rather than follow a parabolic or hyperbolic trajectory in space? What physical explanation, in terms of the basic dimensions of mass, position and time, can be given for why it is that the eccentricity of a planetary orbit lies between 0 and 1? The answer comes from what is known as *the equation of energy*, to which we now turn.

We saw earlier (see the last of the four boxed equations above) that for motion under a central force

$$\dot{r}^2 + r^2 \cdot \dot{\theta}^2 = h^2 \cdot \left[\left(\frac{du}{d\theta} \right)^2 + u^2 \right].$$

The expression on the left-hand side is the square of the velocity (v^2). Now kinetic energy (i.e., energy due to motion) is $\frac{1}{2} \cdot m \cdot v^2$, whence *kinetic energy per unit mass* (call it T) is $\frac{1}{2} \cdot v^2$. That is,

$$T = \frac{1}{2} \cdot h^2 \cdot \left[\left(\frac{du}{d\theta} \right)^2 + u^2 \right].$$

Let E the *total energy per unit mass*, and let V be *potential energy per unit mass*. We have the basic equation

$$E = T + V, \qquad \text{or} \qquad T = E - V.$$

Hence

$$E - V = \frac{1}{2} \cdot h^2 \cdot \left[\left(\frac{du}{d\theta} \right)^2 + u^2 \right],$$

solution. This is because, when one formulates matters in a logically rigorous fashion, *there is no need* for the integral sign. In fact, its use is *logically misleading*, because indefinite integrals $\int x(t)dt$ masquerade as *singular terms*. But the fundamental theorem itself is a guarantee that, before determining a value for the so-called 'constant of integration', an indefinite integral *does not have a unique denotation*. And this defect is not to be explained away as arising from its role as a parameter within a proof (such as a parameter for $\forall$-Introduction or a parameter for $\exists$-Elimination in natural deduction).

from which we obtain, for motion under a central force, the

$$\text{Equation of Energy} \qquad \left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] = \frac{2 \cdot (E - V)}{h^2}.$$

We must bear in mind also that

$$F_O = \frac{dV}{dr},$$

where F_O is the inward radial component of force. This makes V a function of r, hence also of u. Recall that

$$F_O = \frac{\mu}{r^2}$$

where μ is a constant. Hence

$$F_O = \frac{d}{dr}\left(-\frac{\mu}{r}\right).$$

By the fundamental theorem on integration, we conclude that

$$V = -\frac{\mu}{r} + a$$

for some constant a. It is convenient to choose $a = 0$, so that V tends to 0 as r tends to infinity. Henceforward, then, we have, for motion under a central force,

$$V = -\frac{\mu}{r} = -\mu \cdot u.$$

We are now equipped with the means to determine the orbit's **eccentricity** and **semi-latus rectum** in physical terms. Recall our earlier equation for motion under a central *attractive inverse-square* force:

$$u(\theta) = a \cdot \cos(\theta + b) + \frac{\mu}{h^2}$$

and consider it now along with the equation of energy:

$$\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] = \frac{2 \cdot (E - V)}{h^2}.$$

We want to determine the constants a and b in terms of the physical magnitudes E (total energy per unit mass) and h (angular momentum per unit mass). Using our expression for $u(\theta)$, we calculate the square of its first derivative (with respect to θ), and its square:

$$\left(\frac{du}{d\theta}\right)^2 = \left(\frac{d}{d\theta}\left(a\cdot\cos(\theta+b)+\frac{\mu}{h^2}\right)\right)^2 = a^2\cdot(-\sin(\theta+b))^2 \;=\; a^2\cdot(\sin(\theta+b))^2$$

and

$$u^2 = \left(a\cdot\cos(\theta+b)+\frac{\mu}{h^2}\right)^2 \;=\; a^2\cdot(\cos(\theta+b))^2 + 2\cdot a\cdot\frac{\mu}{h^2}\cdot\cos(\theta+b)+\frac{\mu^2}{h^4}$$

whence the left-hand side of the equation of energy has been developed as follows:

$$\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] = a^2 + 2\cdot a\cdot\frac{\mu}{h^2}\cdot\cos(\theta+b)+\frac{\mu^2}{h^4}.$$

The right-hand side of the equation of energy can be developed, in its turn, as follows:

$$\frac{2\cdot(E-V)}{h^2} = \frac{2\cdot(E-(-\mu\cdot u))}{h^2} = \frac{2\cdot(E+\mu\cdot u)}{h^2}$$

$$= \frac{2\cdot(E+\mu\cdot(a\cdot\cos(\theta+b)+\frac{\mu}{h^2}))}{h^2} = \frac{2\cdot E}{h^2} + 2\cdot a\cdot\frac{\mu}{h^2}\cdot\cos(\theta+b) + 2\cdot\frac{\mu^2}{h^4}.$$

Since the term $2\cdot a\cdot\frac{\mu}{h^2}\cdot\cos(\theta+b)$ occurs in the development of each side of the energy equation, it may be dropped, and the equation resulting from the two developments is

$$a^2 + \frac{\mu^2}{h^4} = \frac{2\cdot E}{h^2} + 2\cdot\frac{\mu^2}{h^4}, \qquad \text{whence} \qquad a^2 = \frac{2\cdot E}{h^2} + \frac{\mu^2}{h^4}.$$

From this it follows easily that

$$a = \frac{\mu}{h^2}\sqrt{1+\frac{2\cdot E\cdot h^2}{\mu^2}}.$$

As was pointed out earlier, the constant b enters only as an accommodation of different choices of reference line for the determination of the angle θ in polar coordinates. So we can choose the value 0 for b, simply by settling on a reference line that makes it so. Henceforth all expressions of the form $\cos(\theta+b)$ can be replaced with $\cos\theta$. Thus the equation for motion under a central attractive inverse-square force becomes

$$u(\theta) = a\cdot\cos\theta + \frac{\mu}{h^2}.$$

Hence

$$u(\theta) = \frac{\mu}{h^2} \cdot \left(1 + \frac{h^2}{\mu} \cdot a \cdot \cos\theta\right).$$

Substituting the expression just derived above for a, we obtain

$$u(\theta) = \frac{\mu}{h^2} \cdot \left(1 + \frac{h^2}{\mu} \cdot \frac{\mu}{h^2}\sqrt{1 + \frac{2 \cdot E \cdot h^2}{\mu^2}} \cdot \cos\theta\right).$$

Thus

$$u(\theta) = \frac{\mu}{h^2} \cdot \left(1 + \sqrt{1 + \frac{2 \cdot E \cdot h^2}{\mu^2}} \cdot \cos\theta\right).$$

Recall our earlier expression for a conic in terms of its eccentricity e and semi-latus rectum λ (but here with b conveniently set to 0):

$$\boxed{u = \tfrac{1}{\lambda} \cdot (1 + e \cdot \cos\theta)}$$

It follows that with a point-particle moving under a central, attractive, inverse-square force, its path, which we already know to be a conic, will have

$$\text{eccentricity } e = \sqrt{1 + \frac{2 \cdot E \cdot h^2}{\mu^2}}, \text{ and semi-latus rectum } \lambda = \frac{h^2}{\mu}.$$

The eccentricity e is non-negative by definition. For the conic in question to be an ellipse, we need moreover $e < 1$. The eccentricity will be exactly 0 —and the ellipse in question a circle—just in case $E = -\frac{\mu^2}{2 \cdot h^2}$. That is one extreme special case of an elliptical orbit. Inspection of the expression for e shows that $e < 1$ (whence the orbit is an ellipse) just in case $E < 0$. For the orbit to be a parabola, we need $E = 0$; and for an hyperbola, $E > 0$. Thus it is the *total energy* (kinetic plus potential) of the particle that determines whether its conic path is an ellipse, a parabola, or an hyperbola. The parabola is a limiting case between an ellipse and an hyperbola. These two non-elliptical conics are reserved for those bodies that are energetic enough to be able to move 'off to infinity' despite the influence of the central body's gravitational attraction. Those that are not energetic enough will find themselves in elliptical orbits around the central body.

So, in the case of a planet in an elliptical orbit, we are making the hypothesis that its total energy E is less than 0. We can call this a 'boundary-condition' assumption, and advert to it as BC in subsequent diagrams.

2.4.5. Kepler's Third Law

It remains to furnish a Newtonian explanation of **Kepler's Third Law**. This requires a characterization of the planetary year, or *periodic time*.

We saw earlier that the areal speed is constant:

$$\dot{A} = \frac{h}{2}.$$

Just as time is distance divided by speed, so too time is area divided by areal speed. So the time taken for the radius to sweep out a given area A is governed by the following equation:

$$\text{time taken to sweep out area } A = \frac{2 \cdot A}{h}.$$

The periodic time τ is the time taken to sweep out the entire area of the ellipse. The area of the ellipse is $\pi \cdot \alpha \cdot \beta$, where α is the semi-major axis and β is the semi-minor axis. Thus

$$\text{periodic time } \tau = \frac{2 \cdot \pi \cdot \alpha \cdot \beta}{h}.$$

The two semi-axes are related by the formula[11]

$$\beta = \alpha \cdot \sqrt{1 - e^2}.$$

Hence

$$\tau = \frac{2 \cdot \pi \cdot \alpha \cdot \alpha \cdot \sqrt{1 - e^2}}{h}.$$

So

$$\tau^2 = \frac{4 \cdot \pi^2 \cdot \alpha^4 \cdot (1 - e^2)}{h^2}.$$

Now earlier we proved the geometric fact that the semi-latus rectum of an ellipse is $\alpha \cdot (1 - e^2)$. (This was the conclusion of the 'mushroom cloud' above.) We have also derived the dynamical characterization of the semi-latus rectum of an elliptical orbit as h^2/μ. So we have the dynamical equation

$$\frac{h^2}{\mu} = \alpha \cdot (1 - e^2).$$

[11] This formula, and the one for the area of an ellipse, are the only geometrical claims whose proof is left to the reader.

From this it follows that

$$h^2 = \mu \cdot \alpha \cdot (1 - e^2).$$

Hence, substituting in the last equation for τ^2, we obtain

$$\tau^2 = \frac{4 \cdot \pi^2}{\mu} \cdot \alpha^3.$$

This tells us that the square of the planetary year τ is proportional to the cube of the semi-major axis α of the planetary orbit. Moreover, the proportionality factor $4 \cdot \pi^2/\mu$ involves only constants that are independent of the planet in question. (Remember that $\mu = G \cdot M$, where G is the gravitational constant and M is the mass of the Sun.) Thus the ratio τ^2/α^3 is *the same for all planets*. **Kepler's Third Law** is now explained.

3. Philosophical and methodological discussion

We now have the sought explanation, in physical terms, of why it is that certain heavenly bodies—the planets—have elliptical orbits around the Sun. They are the ones that are not energetic enough to escape the Sun's gravitational field. We have derived Kepler's First Law of planetary motion, as an *explanandum*, from two high-level scientific hypotheses (**Newton's Second Law of Motion**, and his inverse-square **Law of Gravitation**) and the further boundary-condition assumption that planetary energies are not high enough for them to move arbitrarily far away from the Sun.

That the *only* such orbits are ellipses is a remarkable result. The hypotheses in question (including the boundary-condition assumption) rule out the possibility of, say, an orbit that spirals inwards until the orbiting body collides with the Sun. Even if we are aware that both the Sun and the planet are idealized as mass-*points*, with no spatial extension, this is an arresting result. With less idealization, one could treat both bodies as spheres of uniform density, and assume that at some point in the planet's orbit the minimum distance between the surfaces of the two bodies is positive. Even then—with the prospect of a collision not being 'geometrically far-fetched'—Newton's theory tells us that the planet, while it moves under the influence of the Sun's gravity alone, will stay at least that minimum distance away from the Sun.

Note, however, that we have not yet shown how to use Newtonian theory to make predictions concerning the future whereabouts of orbiting bodies, on the basis of knowledge of their current positions and velocities.

But those sorts of precise spatio-temporal predictions, while certainly *useful* in confirming the theory, are not *necessary*. For the order of historical discovery might have been very different. It might have been the case that Newton put forward his general theory of motion and of gravitation well before anyone (like Kepler) arrived at lower-level generalizations about planetary orbits. Armed with the foregoing *a priori* deductions, the theorist would have had **Kepler's First Law** (and the more general result about mass-particles in a central gravitational field tracing conics) as a *prediction* of the general theory. Astronomical observations could then have been made in order to test the prediction. If the observational data revealed that the orbits were indeed elliptical, with the Sun at one focus, then the theory (along with the boundary-condition assumption that for each planet p, $E_p < 0$) would be empirically confirmed.

As it happens, Kepler's three laws were formulated before Newton's theory. But this does not alter the *logical* analysis on offer of how Newton's theory can be empirically tested. Let us return to our imagined scenario in which Newton's theory was available before Kepler's First Law. And let us suppose (again, contrary to fact) that planetary orbits, upon analysis of the observational data, turned out *not* to be elliptical, even though they were closed. Then the logical structure of this supposed empirical refutation of Newton's theory would be as follows:

$$\begin{array}{l}
\text{Mathematical axioms} \\
\qquad\qquad\vdots \\
\underbrace{\text{Mathematical theorems , Scientific hypotheses , BCs}} \\
\qquad\qquad\qquad\qquad\qquad\vdots \\
\qquad\qquad\underbrace{\text{Prediction (Kepler's First Law) , Observations/measurements}} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\vdots \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\bot
\end{array}$$

Here, the statements of observation and measurement feature as *premises* in a *disproof* (that is, a proof whose conclusion is the absurdity symbol $\bot$), and accordingly occupy a new branch-tip within the tree. Such a disproof establishes the impossibility of the joint truth of the mathematical axioms, scientific hypotheses, boundary-condition assumptions, and observations and measurement. Usually, the mathematical axioms are taken to be *a priori*, and are not questioned. Consequently, in one's response to the contradiction, one seeks to find fault with at least one of the remaining premises:

the scientific hypotheses, the boundary-condition assumptions (here, the assumption that $E < 0$) or the statements of observation and measurement of planetary positions.

All the reasoning involved in arriving at an answer to our question about the motion of a body under the influence of a central, attractive, inverse-square force was *strictly deductive*. Nowhere did we make inferential transitions from finite data to claims of a general kind. We took no inductive steps. That is not to say that the same would hold for the way in which *Newton* arrived at his laws of motion and his law of universal gravitation. Newton himself might have 'derived' his laws inductively, drawing on his experience. Alternatively, he might have 'inferred to the best explanation', by canons different from those of deductive logic. But it is not for us to decide this matter here. Rather, we want to understand the logical structure of predictions and explanations that are made possible by those very laws, *after* they have been formulated and put forward as scientific hypotheses. The case of motion under gravitational attraction has been instructive in this regard.

What we have seen is a structure of deductive proof taking the following form, where the direction of logical deduction is downwards:

$$\begin{array}{c} \begin{array}{l} \text{Mathematical axioms} \\ \qquad\quad \vdots \end{array} \\ \underbrace{\text{Mathematical theorems} \quad , \quad \text{Scientific hypotheses of great generality}} \\ \vdots \\ \text{Lower-level law about a certain class of phenomena} \end{array}$$

The claim that motion under gravitational attraction will trace a conic (the 'lower-level law' in question) was the *conclusion* of a scientific argument, or proof. In the foregoing display, the direction of logical inference is to be understood as *downwards*. The *premises* for the deduction are, ultimately, the mathematical axioms and the scientific hypotheses. In this case, no use is made of any assumptions concerning boundary or initial conditions.[12] The premises mentioned occupy tips of the branches of the foregoing 'proof tree'. The conclusion—here, the lower-level law about motion on conics—

[12] In the case of a planet in an elliptical obit—as we shall see below—we shall only require, in addition, the boundary-condition assumption that $E < 0$, and certain *initial* conditions in order to generate predictions as to the future whereabouts of a planet whose present position and velocity are known.

is placed at the root. The proof is deductive, hence logically valid. *If* its premises are true, *then* its conclusion is true also. Concerning that inferential transition, there is no possible doubt.

We saw that the mathematical theorems involved (in the case of a planetary orbit) were quite simple. They included the fundamental theorem on integration, and various quite accessible theorems about derivatives of linear, quadratic and trigonometric functions. Several kinematic theorems were used that exploited the convenience afforded by a system of polar coordinates.[13] We also appealed to geometrical axioms in order to prove useful theorems about ellipses.

The scientific hypotheses were **Newton's First and Second Laws of Motion**, and his **Law of Universal Gravitation**. We also appealed to the boundary-condition assumption that the total energy E of a planet is negative. From the logician's point of view, this too is just as 'hypothetical' as the **Laws** just mentioned. It is one of the undischarged premises from which our deductive reasoning proceeds.

We are now in a position to see why this model of scientific explanation is called *hypothetico-deductive*. It is because the *scientific hypotheses* and the *boundary-condition assumption* (about energy) are *premises* of a proof, which is fully *deductive*, and which has the explanandum *as its conclusion*.

The mathematics 'connects' with the physics as follows. Time is treated as a continuum of temporal instants isomorphic to the real numbers. (One can correlate 'now' with 0, 'earlier than' with $<$, the future with the positive reals, the past with the negative reals, etc.) Physical space is treated as a three-dimensional manifold (although the problem of planetary orbits was discussed and solved in the more restrictive context of a plane, i.e. a two-dimensional manifold). Time-dependent coordinates of position (of a particle) are represented as continuous functions from the reals to the reals. (The coordinates r and θ were like this.) Moreover, certain physical magnitudes were assumed to take real numbers as values (with reference to an appropriately chosen *unit* of measurement for the physical magnitude in question). That unit of measurement would correlate with the real number 1, and other values of the magnitude in question would be correlated 1-1 with the remaining reals. (The magnitudes m, F_O, E and h were like this.)

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[13] The applied mathematician makes insightful choices of coordinate systems that are designed to make the mathematical reasoning as straightforward as possible. In the context of a different problem, such as the Newtonian explanation of the trajectory of a projectile, it is more convenient to use rectangular Cartesian coordinates rather than the polar coordinates chosen here.

Certain scientific hypotheses are stated as differential equations governing time-dependent, real-valued (or vector-valued) physical magnitudes. This is what enables one to *apply* the mathematics to the physical problem-context. Mathematics supplies theorems furnishing solutions to differential equations of certain forms. These solutions then find physical interpretations in terms of the time-dependent physical magnitudes in question. Thus one can ponder a physical problem, represent its important features in mathematical terms, prove mathematical theorems involving those representations, and use the theorems to translate results back into purely physical terms. The excursus into the mathematics effects a great deal of deductive compression, and becomes practically indispensable, despite any arguments that might be advanced for the 'in-principle eliminability' of the mathematical discourse itself.

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References

[1] Gilder, Joshua, and Anne-Lee Gilder, *Heavenly Intrigue: Johannes Kepler, Tycho Brahe, and the Murder Behind One of History's Greatest Scientific Discoveries*, Doubleday, 2004.

[2] Newton, Isaac, *The Principia: Mathematical Principles of Natural Philosophy*, A New Translation by I. Bernard Cohen and Anne Whitman, University of California Press, Berkeley, CA, 1999.

[3] Synge, John L., and Byron A. Griffith, *Principles of Mechanics, 3rd edition*, McGraw-Hill, New York, 1959.

[4] Tennant, Neil, *Anti-Realism and Logic: Truth as Eternal*, Clarendon Library of Logic and Philosophy, Oxford University Press, 1987.

[5] Tennant, Neil, 'Natural foundations for synthetic projective geometry', *Paper presented to the 8th Annual Midwest Philosophy of Mathematics Workshop at Notre Dame in October 2007*, (2008).

[6] Tennant, Neil, 'Natural logicism via the logic of orderly pairing', in Sten Lindström, Erik Palmgren, Krister Segerberg, and Viggo Stoltenberg-Hansen, (eds.), *Logicism, Intuitionism, Formalism: What has become of them?*, Synthese Library, Springer Verlag, 2008, pp. 91–125.

[7] Tennant, Neil, 'Natural Logicist Foundations for Functions of Reals and their Derivatives', *Unpublished typescript*, (2008).

[8] Tennant, Neil, 'A Pair of Paradoxical Counterfactuals Apparently Supported by Newton's Second Law of Motion and Law of Gravitation; or, Why the Third Law of Motion is Needed, but is a Ladder that can be Kicked Away', *Unpublished typescript*, (2008).

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