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FREGE'S CONTENT-PRINCIPLE AND RELEVANT DEDUCIBILITY

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ABSTRACT. Given the harmony principle for logical operators, compositionality ought to ensure that harmony should obtain at the level of whole contents. That is, the role of a content *qua* premise ought to be balanced exactly by its role as a conclusion. Frege's contextual definition of propositional content happens to exploit this balance, and one appeals to the Cut rule to show that the definition is adequate.

We show here that Frege's definition remains adequate even when one relevantizes logic by abandoning an unrestricted Cut rule. The proof exploits the fact that in the relevantized logic, which abandons the unrestricted rule of Cut, any failure of the transitivity of deduction is offset by the epistemic gain involved in learning that a stronger-than-expected result holds.

KEY WORDS: content, cut, entailment, harmony, intuitionistic logic, perfect validity, relevant logic, transitivity of deduction

1. AIM

The aim of this paper is to show that Frege's contextual definition of identity of propositional content (in terms of premise-equivalence) is ideally suited to relevance logics in which Cut does not hold unrestrictedly, but does hold in the restricted form 'If $X \vdash P$ and $Y, P \vdash Q$ then for some subset Z of $X \cup Y$, either $Z \vdash Q$ or $Z \vdash \perp$ '.

2. FREGE'S BEGRIFFLICHER INHALT

In §3 of his *Begriffsschrift*, Frege enunciated the following principle concerning propositional content (*begrifflicher Inhalt*):

the propositional contents of two judgments can differ in two ways: first, it may be the case that the consequences (*Folgerungen*) that can be inferred (*gezogen werden können*) from one of them in conjunction with certain others, always follow (*folgen*) also from the other in conjunction with those other judgments; secondly, this may not be the case.

In the first case, he called

that part of the content, which is the same in both [of the judgments], the *propositional content*.



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This might well be the earliest contextual definition that Frege gave. We shall speak of sentences, where he spoke of judgments; and we shall abbreviate ‘propositional content’ to ‘content’. Given a sentence P , its content will be denoted by $\S P$. Let X, Y be sets of sentences, and let \vdash indicate deducibility – the relation answering to the verbs ‘gezogen werden können’ and ‘folgen’, which Frege uses interchangeably. Let us use the abbreviation X, P for $X \cup \{P\}$. The complex variable X_{PQ} will be understood as ranging over sets containing neither P nor Q . The contextual definition in question can then be rendered as follows.

$$\S P = \S Q \stackrel{\text{df}}{\equiv} \forall X_{PQ} \forall R (X_{PQ}, P \vdash R \Leftrightarrow X_{PQ}, Q \vdash R).$$

We shall call this the *Premise Principle for Content Identity*. It says that two contents are identical just in case the respective sentences that have them can replace other in premise-position in any valid argument, *salva validitate* – in other words, just in case those sentences are *premise-equivalent*. In what follows, we shall suppress the subscript PQ , and let the intended restriction on the range of the variable X in any version of premise-equivalence be implicitly understood from the context.

As Frege went on to express the matter, premise-equivalence suffices for content-identity because, in his formalized language,

with a judgment ... the only thing taken into consideration ... is that which has an influence on its *possible consequences*. (*weil im Urtheile ... nur das in Betracht kommt, was auf die möglichen Folgerungen Einfluss hat.*)

The identity of a content, then, is exhausted (and fixed) by those *logically downstream contents* that flow from it (in conjunction with various other contents). Note that this is very much like saying, of a logical operator, that its sense is exhausted (and fixed) by the rule for its elimination in a major premise. Indeed, premise-conclusion harmony is closely linked to the kind of harmony that obtains between the introduction rule for a logical operator, and its corresponding elimination rule. The introduction rule governs inference *to* a conclusion, and the elimination rule inference *from* a major premise, with a dominant occurrence of the operator in question. Premise-conclusion harmony could be thought of as the global result, at the level of whole contents, of the local equilibrations effected by harmonious rules of introduction and elimination for the logical constituents of those contents.¹

Among the valid arguments under consideration when abstracting the content of a given sentence will be *reductios*, whose conclusions are \perp . That is to say, the variable R in the Premise Principle includes \perp in its range – which we shall count as a formal consequence of inconsistent sets of premises. Whenever it appears as the conclusion of a sequent, however – as in $X : \perp$ – we should construe \perp as the empty set of sentences. In this

way we can say that $X : \perp$ is a subsequent of $X : Y$ (and, equivalently, that $X : Y$ is a supersequent of $X : \perp$).

Why did Frege think it sufficed to concentrate on the role played by a judgment as a *premise* in arguments, when fixing on its content? What about the role it also plays as a *conclusion* of arguments? Might that not contribute some extra ingredient to its content? The answer is negative. Call P and Q *conclusion-equivalent* just in case

$$\forall X(X \vdash P \Leftrightarrow X \vdash Q),$$

where X is understood as ranging over all sets of sentences, whether or not they contain P or contain Q .

ADEQUACY THEOREM. *Premise-equivalent sentences are conclusion-equivalent, and vice versa.*

Proof. Assume P and Q are premise-equivalent. We aim to show that P and Q are conclusion-equivalent.

First we show that premise-equivalent sentences are interdeducible. To this end we note the proof

$$\frac{\frac{\frac{\forall X \forall R(X, P \vdash R \Leftrightarrow X, Q \vdash R)}{\forall R(P \vdash R \Leftrightarrow Q \vdash R)}}{P \vdash Q \Leftrightarrow Q \vdash P} \quad Q \vdash Q}{P \vdash Q}$$

and its obvious companion for the converse deducibility statement $Q \vdash P$.

Let X be an arbitrary set of sentences. Suppose that $X \vdash P$. Since $P \vdash Q$, we have, by transitivity of \vdash (Cut), that $X \vdash Q$. The argument for the converse (from $X \vdash Q$ to $X \vdash P$) is similar. So $X \vdash P \Leftrightarrow X \vdash Q$. Since X was arbitrary, it follows that P and Q are conclusion-equivalent.

For the 'vice versa' part, assume that P and Q are conclusion-equivalent. We aim to show that they are premise-equivalent. First we show that conclusion-equivalent sentences are interdeducible. To this end we note the proof

$$\frac{\frac{\forall X(X \vdash P \Leftrightarrow X \vdash Q)}{P \vdash P \Leftrightarrow P \vdash Q}}{P \vdash Q}$$

and its obvious companion for the converse deducibility statement $Q \vdash P$.

Let X be an arbitrary set of sentences containing neither P nor Q , and let R be an arbitrary sentence. Suppose that $X, P \vdash R$. Since $Q \vdash P$, we have, by transitivity of \vdash (Cut), that $X, Q \vdash R$. The argument for the

converse (from $X, Q \vdash R$ to $X, P \vdash R$) is similar. So $X, P \vdash R \Leftrightarrow X, Q \vdash R$. Since X and R were arbitrary choices of the specified kinds, it follows that P and Q are premise-equivalent. \square

Note that no assumptions were made about the expressive power of the language. No particular logical operators need to be in the language in order for the argument to go through. Nor was any classical or realist assumption made about the language being bivalent. The foregoing argument is just as convincing to the intuitionist (anti-realist) as it is to the classicist (realist). Moreover, the result it establishes can be appreciated, with hindsight, as *of the essence* of propositional identity: that whatever is required in order to be able to *infer to* a proposition should be exactly matched, in content-determining strength, by whatever is permitted in *inferring from* the same. We can call this the ideal of *premise-conclusion harmony*.

THEOREM. *The interdeducibility of P and Q is not only a necessary condition, but also a sufficient condition for the identity of the contents $\S P$ and $\S Q$.*

Proof. If P and Q are interdeducible, then from $X, P \vdash R$ it will follow (by Cut) that $X, Q \vdash R$, and vice versa:

$$\frac{Q \vdash P \quad X, P \vdash R}{X, Q \vdash R} \quad \frac{P \vdash Q \quad X, Q \vdash R}{X, P \vdash R} \quad \square$$

It was an avoidable quirk on Frege's part that he chose to reify contents, and to entertain claims such as $\S P = \S Q$, in which singular terms purport to refer to contents as abstract objects. One could simply speak instead of P and Q being synonymous, and express the matter relationally, with a notation such as $P \cong Q$. The upshot is that given the latter as a definiendum, there are two equivalent definientia: that P and Q are premise-equivalent; and that P and Q are conclusion-equivalent.

3. TWO DEVIATIONS FROM CLASSICAL LOGIC: INTUITIONIZING AND RELEVANTIZING

When Frege wrote *Begriffsschrift*, thereby founding modern classical logic, the deviant subsystem of intuitionistic logic, due later to Brouwer and Heyting, had not been identified, let alone given any philosophical justification. Frege could hardly have anticipated that later exponents of intuitionism, such as Dummett, would base their philosophical justifications for the choice of intuitionistic logic as the correct logic on considerations

of meaning that would pay very close attention to the harmoniously balanced roles that sentences can play both as major premisees of inferences and as their conclusions.

As the first modern classical logician, Frege was also ‘non-relevantist’ and ‘non-paraconsistent’ in his conception of logical consequence (and logical deducibility). That is to say, he endorsed the inference $A, \neg A : B$, now known as Lewis’s first paradox, which both relevant and paraconsistent logicians wish to avoid. Frege could hardly have anticipated that later exponents of relevant logic, such as Anderson and Belnap, would base their philosophical justifications for the choice of a relevant logic as the correct logic on considerations of ‘meaning-connections’ that would pay very close attention to the role of undischarged assumptions of proofs, and the notion of an assumption having been genuinely *used* in order to derive a conclusion.

Both intuitionistic and relevantist reforms have been advocated since Frege’s time. But while there is universal agreement among logicians on how, formally, to ‘intuitionize’ classical first-order logic, there are at least two distinct approaches to the project of ‘relevantizing’.

4. TWO APPROACHES TO RELEVANTIZING

The first approach to relevantizing, due to Anderson and Belnap, is to avoid Lewis’s first paradox by giving up disjunctive syllogism

$$A \vee B, \neg A : B$$

while retaining unrestricted transitivity of deduction, in the form of the well-known rule of Cut:

$$\frac{X \vdash C \quad C, Y \vdash B}{X, Y \vdash B}.$$

This enables one to avoid the erstwhile proof

$$\frac{\frac{A \vdash A}{A \vdash A \vee B} \quad A \vee B, \neg A \vdash B}{A, \neg A \vdash B}_{\text{CUT}}$$

of Lewis’s first paradox. On this approach, the burden of relevantizing is borne more by changes in the behavior of the connectives of the object-language, than by changes in the structure of the deducibility relation.

In the context of our current concerns, we note that the relevantist in the Anderson–Belnap tradition can take over exactly the same Fregean treatment of content-identity explained above. The rule of Cut will guarantee

that premise-equivalence will be equivalent to conclusion-equivalence, so that the ideal of premise-conclusion harmony is met.

It is far from obvious that the same will hold of the second approach to relevantizing.² On this approach, one retains disjunctive syllogism but avoids the proof just given by not endorsing the *unrestricted* rule of Cut. Cuts are not permitted; so all relevant proofs are cut-free. But a restricted form of Cut, which we shall call (RC), is nevertheless admissible for the preferred system of relevant logic:

(RC) If there is a relevant proof of $X : C$ and there is a relevant proof of $Y, C : B$ then there is a relevant proof of some *subsequent* of $X, Y : B$.

For example, there is a relevant proof of $A : A \vee B$, and there is a relevant proof of $A \vee B, \neg A : B$. There is also a relevant proof of $A, \neg A : \perp$ – thus vindicating (RC) – even though there is no relevant proof of its supersequent $A, \neg A : B$.

(RC) is a corollary of a more general and powerful principle about the transformability of proofs. Let S be either the system of classical logic, or the system of intuitionistic logic. Let SR be the corresponding relevantized system.

RELEVANTIZING PRINCIPLE (RP). *Any S -proof whose conclusion is A and whose undischarged assumptions form the set X , can be effectively transformed into an SR -proof whose conclusion is either A or \perp , and whose undischarged assumptions are in X .*

Thus relevantizing a proof either yields a proof of the original result, or yields a *stronger* result that gives us *epistemic gain*. (One is always better off knowing that a proper subsequent of a given sequent is valid, rather than just knowing that the latter is valid.)

DEFINITION. A sequent is *perfectly valid* (in a given system) just in case it is valid and contains no proper subsequent that is valid in the system.

COROLLARY TO RP. *If a sequent is perfectly valid in the non-relevant parent system S , then it is provable in the relevantized system SR .*

On this approach, the burden of relevantizing is borne more by changes in the structure of the deducibility relation. The connectives retain, in so far as this is possible, their original (classical or intuitionistic) senses. This is borne out by the corollary.

To summarize the two approaches to relevant logic: the method chosen by Anderson and Belnap was to give up disjunctive syllogism but

to hold on to unrestricted transitivity of deduction; whereas the method that I favour is to retain disjunctive syllogism but to restrict transitivity of deduction to where it really matters. This is not the place to argue in favour of this method of relevantizing over that of Anderson and Belnap. The project in this paper is to assume this method of relevantizing as preferred, and to proceed to resolve the problem foreshadowed above for the attempt to adopt Frege's conception of, and criterion for, identity of propositional content.

5. FREGE'S NOTION OF PROPOSITIONAL CONTENT IN THE RELEVANTIZED SETTING

The problem arises for the earlier characterization of propositional content because we are now relevantizing by giving up Cut, or unrestricted transitivity of deduction. Yet the foregoing proof of the equivalence of premise-equivalence and conclusion-equivalence made important appeals to the rule of Cut. Can the reasoning be rescued in the meta-context of our method of relevantizing? Can we uphold the ideal of premise-conclusion harmony in the relevantized setting?

The answer is affirmative. In all subsequent reasoning, we let \vdash be the deducibility relation of the *relevantized* system of logic.

DEFINITION. $X \overset{\square}{\vdash} A \equiv_{df} X \vdash A$ and no proper subsequent of $X : A$ is provable. We shall read ' $X \overset{\square}{\vdash} A$ ' as ' $X : A$ is *perfectly* (relevantly) provable'.

OBSERVATION 1. For any sentence A , exactly one of the following holds:

- (i) A is logically true, i.e. $\overset{\square}{\vdash} A$;
- (ii) A is logically false, i.e. $A \overset{\square}{\vdash} \perp$;
- (iii) A is contingent, i.e. $\not\overset{\square}{\vdash} A$ and $A \not\overset{\square}{\vdash} \perp$.

We shall follow Casimir Lewy in calling (i), (ii) and (iii) the three possible *modal values* of A .

OBSERVATION 2. For any contingent sentence A , we have $A \overset{\square}{\vdash} A$.

We now replace the notion of deducibility in the equivalence concepts above with the notion of perfect (relevant) deducibility. Thus the claim that P and Q are premise-equivalent becomes

$$\forall X \forall R (X, P \overset{\square}{\vdash} R \Leftrightarrow X, Q \overset{\square}{\vdash} R),$$

where once again it is understood that X is restricted so as to range over sets of sentences containing neither P nor Q .

Likewise the claim that P and Q are conclusion-equivalent becomes

$$\forall X (X \vdash^{\square} P \Leftrightarrow X \vdash^{\square} Q),$$

where there is no need for any corresponding restriction on the range of X .

LEMMA 1. *Premise-equivalent sentences have the same modal value.*

Proof. We show that each of the three ways that any two sentences can fail to enjoy the same modal value would contradict premise-equivalence. These three ways are:

- (1) one of the sentences is logically true, while the other is contingent;
- (2) one of the sentences is logically false, while the other is contingent;
- and
- (3) one of the sentences is logically true, while the other is logically false.

Suppose A and B are premise-equivalent. Since A and B are arbitrary, there is no loss of generality in taking A as the first-mentioned sentence in each case, and B as the other sentence.

Case (1): (a) $\vdash^{\square} A$, (b) $\not\vdash^{\square} B$ and (c) $B \not\vdash^{\square} \perp$. Suppose $A \vdash^{\square} B$. From this and (a), it would follow by (RC) that $\vdash^{\square} B$, contrary to (b). Hence $A \not\vdash^{\square} B$. But by Observation 2, we have $B \vdash^{\square} B$. This contradicts the premise-equivalence of A and B .

Case (2): (a) $A \vdash^{\square} \perp$, (b) $\not\vdash^{\square} B$ and (c) $B \not\vdash^{\square} \perp$. From (a) it follows by the premise-equivalence of A and B that $B \vdash^{\square} \perp$, contradicting (c).

Case (3): (a) $\vdash^{\square} A$ and (b) $B \vdash^{\square} \perp$. By premise-equivalence of A and B , we have from (b) that $A \vdash^{\square} \perp$. Hence from (a), by (RC), we would have $\vdash^{\square} \perp$, which is impossible.

LEMMA 2. *Conclusion-equivalent sentences have the same modal value.*

Proof. The strategy of proof is exactly the same as for the previous lemma. We investigate the same three cases as we did there.

Case (1): (a) $\vdash^{\square} A$, (b) $\not\vdash^{\square} B$ and (c) $B \not\vdash^{\square} \perp$. From (a) it would follow by the conclusion-equivalence of A and B that $\vdash^{\square} B$, contradicting (b).

Case (2): (a) $A \vdash^{\square} \perp$, (b) $\not\vdash^{\square} B$ and (c) $B \not\vdash^{\square} \perp$. Suppose $B \vdash^{\square} A$. Then from (a) and by (RC) we would have $B \vdash^{\square} \perp$, contrary to (c). Hence $B \not\vdash^{\square} A$.

But by Observation 2, we have $B \vdash^{\square} B$. This contradicts the conclusion-equivalence of A and B .

Case (3): (a) $\vdash^{\square} A$ and (b) $B \vdash^{\square} \perp$. From (a), by the conclusion-equivalence of A and B , we have $\vdash^{\square} B$, whence from (b), by (RC), we would have $\emptyset \vdash^{\square} \perp$, which is impossible.

LEMMA 3. *Logically true sentences are premise-equivalent.*

Proof. No logically true sentence can feature as a premise in any true statement of perfect validity; for it could always be suppressed. Hence, trivially, any two logically true sentences can be intersubstituted for one another, *salva veritate*, in any true statement of perfect validity involving either of them as a premise – for there are no such statements.

LEMMA 4. *Logically true sentences are conclusion-equivalent.*

Proof. Suppose A is logically true. Then the only true statement of perfect validity involving A as a conclusion is the statement $\vdash^{\square} A$. The result is immediate.

LEMMA 5. *Logically false sentences are premise-equivalent.*

Proof. Suppose A is logically false. Then the only true statement of perfect validity involving A as a premise is the statement $A \vdash^{\square} \perp$. The result is immediate.

LEMMA 6. *Logically false sentences are conclusion-equivalent.*

Proof. No logically false sentence can feature as a conclusion in any true statement of perfect validity; for it could always be suppressed in favour of \perp . Hence, trivially, any two logically false sentences can be intersubstituted for one another, *salva veritate*, in any true statement of perfect validity involving either of them as a conclusion – for there are no such statements.

RELEVANTIZED ADEQUACY THEOREM. *Any two sentences are premise-equivalent if and only if they are conclusion-equivalent.*

Proof. Let P and Q be arbitrary sentences. For the ‘only if’ direction, assume P and Q are premise-equivalent. By Lemma 1, P and Q have the same modal value: either they are both logically true, or they are both logically false, or they are both contingent. If they are both logically true,

then by Lemma 4 they are conclusion-equivalent. If they are both logically false, then by Lemma 6 they are conclusion-equivalent. So it remains to investigate the case where they are both contingent. Recall Observation 2:

for any contingent sentence A , we have $A \vdash^{\square} A$. We are assuming that P and Q are contingent, premise-equivalent sentences. We aim to show that P and Q are conclusion-equivalent.

First we show that contingent premise-equivalent sentences are perfectly interdeducible. To this end we note the proof

$$\frac{\frac{\frac{\forall X \forall R (X, P \vdash^{\square} R \Leftrightarrow X, Q \vdash^{\square} R)}{\forall R (P \vdash^{\square} R \Leftrightarrow Q \vdash^{\square} R)}{P \vdash^{\square} Q \Leftrightarrow Q \vdash^{\square} Q} \quad Q \vdash^{\square} Q}{P \vdash^{\square} Q}}$$

and its obvious companion for the converse deducibility statement $Q \vdash^{\square} P$.

Let X be an arbitrary set of sentences. Suppose that $X \vdash^{\square} P$. We need to show that $X \vdash^{\square} Q$. Since $P \vdash^{\square} Q$, we have, by (RC), that for some subset X' of X , either $X' \vdash^{\square} Q$ or $X' \vdash^{\square} \perp$. But $X' \vdash^{\square} \perp$ would contradict $X \vdash^{\square} P$. Hence $X' \vdash^{\square} Q$. Now assume for reductio that X' is a *proper* subset of X . Since $Q \vdash^{\square} P$, we have, by (RC), that for some subset X'' of X' , $X'' \vdash^{\square} P$. But this would contradict the perfect provability of $X : P$. Hence $X' = X$; whence $X \vdash^{\square} Q$. The argument for the converse (from $X \vdash^{\square} Q$ to $X \vdash^{\square} P$) is similar. Since X was arbitrary, it follows that P and Q are conclusion-equivalent.

For the ‘if’ direction, assume that P and Q are conclusion-equivalent sentences. We aim to show that they are premise-equivalent. By Lemma 2, P and Q have the same modal value: either they are both logically true, or they are both logically false, or they are both contingent. If they are both logically true, then by Lemma 3 they are premise-equivalent. If they are both logically false, then by Lemma 5 they are premise-equivalent. So it remains to investigate the case where they are both contingent. We are assuming that P and Q are contingent, conclusion-equivalent sentences. We aim to show that P and Q are premise-equivalent.

First we show that contingent conclusion-equivalent sentences are interdeducible. To this end we note the proof

$$\frac{\frac{P \vdash P \quad \forall X (X \vdash P \Leftrightarrow X \vdash Q)}{P \vdash P \Leftrightarrow P \vdash Q}}{P \vdash Q}$$

and its obvious companion for the converse deducibility statement $Q \vdash P$.

Let X be an arbitrary set of sentences containing neither P nor Q , and let R be an arbitrary sentence. Suppose that $X, P \vdash R$. Since $Q \vdash P$, we have, by (RC), that for some subset X' of $X \cup \{Q\}$, either (1) $X' \vdash R$ or (2) $X' \vdash \perp$. Let X' be arbitrary in this regard. Since $X, P \vdash R$, it follows that Q must be in X' (for otherwise X' would be a subset of X provably implying R or \perp). Thus we can let Y be $X' \setminus \{Q\}$, whence $Y \subseteq X$; and we can set $X' = Y, Q$.

First we show that the second disjunct (2) is impossible. So assume, for reductio, that $X' \vdash \perp$, i.e., that $Y, Q \vdash \perp$. But we have $P \vdash Q$. By (RC), it follows that for some subset Z of $Y \cup \{P\}$, $Z \vdash \perp$. But Y is a subset of X ; whence we have now contradicted $X, P \vdash R$. So $X' \not\vdash \perp$; i.e., $Y, Q \not\vdash \perp$.

So only the first disjunct (1) is possible. That is, we know that $Y, Q \vdash R$, where $Y \subseteq X$. We aim now to show that Y cannot be a *proper* subset of X ; whence it will follow that $X, Q \vdash R$.

So assume for reductio that Y is a proper subset of X . Since $P \vdash Q$, we have by (RC) that for some subset W of $Y \cup \{P\}$, $W \vdash R$ or $W \vdash \perp$. Suppose that W is indeed $Y \cup \{P\}$, so that $Y \cup \{P\} \vdash R$ or $Y \cup \{P\} \vdash \perp$. This would contradict $X, P \vdash R$. Likewise it would contradict $X, P \vdash R$ were W to be a proper subset of $Y \cup \{P\}$. Thus Y cannot be a proper subset of X . Hence $X, Q \vdash R$.

The argument for the converse is similar. Since X and R were arbitrary, it follows that P and Q are premise-equivalent. \square

We have therefore shown that Frege's notion of propositional content is very robust. The form of his definition generalizes over the deviant, proper subsystems of classical logic that interest us, namely intuitionistic and relevant logics.

We saw above that in the classical case the interdeducibility of P and Q was (necessary and) sufficient for their premise-equivalence (equivalently: their conclusion-equivalence). Note that the condition

$$P \vdash Q \text{ and } Q \vdash P$$

is logically simpler than the statement that P and Q are premise-equivalent, which involves universal quantification over X and R . The question now arises whether there is a similar criterial reduction (of premise-equivalence, or, equivalently, of conclusion-equivalence) in the relevantized setting.

DEFINITION. An *entailment* is a substitution instance of a perfectly valid sequent.

OBSERVATION 3. A single example, to be given below, suffices to make the following points.

(1) Even *perfect* (relevant) interdeducibility of P and Q is insufficient to guarantee their interreplaceability, *salva veritate*, in all statements of (relevant) deducibility. *A fortiori*,

(2) the straightforward (relevant) interdeducibility of P and Q is insufficient to the same end; as is their mutual entailment. Finally, since every perfect deducibility is an entailment,

(3) the insufficiencies in question extend to interreplaceability, *salva veritate*, in all statements of entailment.

The example that establishes (1)–(3) is as follows. Note that

$$B \sqsupset (A \wedge \neg A) \vee B \quad \text{and} \quad (A \wedge \neg A) \vee B \sqsupset B.$$

Yet the two sentences B and $(A \wedge \neg A) \vee B$ are *not* interreplaceable, *salva veritate*, in all statements of (relevant) deducibility. More precisely, they are not conclusion-equivalent. To see this, note that

$$A \wedge \neg A \sqsupset (A \wedge \neg A) \vee B, \quad \text{but} \quad A \wedge \neg A \not\sqsupset B.$$

Matters are different, however, when we constrain more carefully the candidate contexts in which replacements are to take place.

THEOREM. If $P \sqsupset Q$ and $Q \sqsupset P$, then P and Q are interreplaceable, *salva veritate*, in all statements of perfect (relevant) deducibility.

Proof. By straightforward application of the line of reasoning used in the proof of the previous theorem, for the case where P and Q are contingent. \square

The reason why the previous example fails to be a counterexample to this theorem is that while

$$A \wedge \neg A \sqsupset (A \wedge \neg A) \vee B,$$

nevertheless

$$A \wedge \neg A \not\vdash^{\square} (A \wedge \neg A) \vee B.$$

This is because the sequent $A \wedge \neg A : (A \wedge \neg A) \vee B$ properly contains the provable subsequent $A \wedge \neg A : \perp$, and therefore cannot be perfectly provable.

It follows from Observation 3 that our last theorem cannot be generalized by replacing perfect (relevant) deducibility throughout by entailment.

6. CONCLUSION

With Frege's original treatment of classical logic, the (non-relevant) interdeducibility of P and Q followed from their premise-equivalence. This was because $P \vdash P$ always holds, regardless of the modal value of P ; whence, by premise-equivalence of P and Q , one has $Q \vdash P$.

When we use perfect deducibility instead of Frege's classical deducibility, the analogue $P \vdash^{\square} P$ of that simple reflexivity principle $P \vdash P$ is available just in case P is contingent. For, if $\vdash P$ then $P \not\vdash^{\square} P$; and likewise if $P \vdash \perp$. It would therefore be a strategic error to seek to identify the content of P (irrespective of its modal value) by appeal to the sentences perfectly interdeducible with P . Only for *contingent* sentences does premise-equivalence (or conclusion-equivalence) guarantee perfect interdeducibility. No two logical truths are perfectly interdeducible; and no two logical falsehoods are perfectly interdeducible. Yet we would like to say that logical truths have the same content – that of absolute understatement; and we would like also to say that logical falsehoods have the same content – that of absolute overstatement. But the relation 'has the same content as' will be confined to contingent sentences if it is based on perfect interdeducibility. By contrast, the relation in question will be defined on all sentences if it is based on premise-equivalence (or, equivalently, on conclusion-equivalence).

It turns out, then, that Frege was uncannily prescient in choosing to give the contextual definition that he did for identity of propositional content. In the classical setting, it would have been so easy simply to use interdeducibility as the definitional criterion for such identity. But Frege did not choose this path. Instead, he chose premise-equivalence as the definitional criterion. In doing so, he chose what proves to be the perfect form for the subsequent extension of that definition to the relevantized setting of perfect deducibility.³

NOTES

¹ The notion of harmony goes back to the seminal writings of Gerhard Gentzen, who laid the foundations for modern proof theory. (See G. Gentzen, 1934–5: ‘Untersuchungen über das logische Schliessen’, *Mathematische Zeitschrift*, I, pp. 176–210; II, pp. 405–431. Translated as ‘Investigations into Logical Deduction’, in *The Collected Papers of Gerhard Gentzen*, edited by M. E. Szabo, North-Holland, Amsterdam, 1969, pp. 68–131.) Gentzen’s great insight was that

every purely logical proof can be reduced to a definite, though not unique, normal form. Perhaps we may express the essential properties of such a normal proof by saying: it is not roundabout. No concepts enter into the proof other than those contained in its final result, and their use was therefore essential to the achievement of that result.

This idea of Gentzen was subsequently called the ‘inversion principle’ by Paul Lorenzen (‘Konstruktive Begründung der Mathematik’, *Mathematische Zeitschrift*, Vol. 53, 1950, pp. 162–202) and by Dag Prawitz (*Natural Deduction: A Proof-Theoretical Study*, Almqvist & Wiksell, Stockholm, 1965). The same idea (in reference to ‘right’ and ‘left’ sequent rules) was stressed by Nuel Belnap in his famous reply to Prior’s paper on the deviant connective ‘tonk’. (See N. Belnap, ‘Tonk, Plonk and Plink’, *Analysis*, Vol. 22, 1962, pp. 130–134.)

In his monograph, Prawitz explicated the harmony of rules formally by means of the *reduction procedures* that would get rid of any sentence-occurrence standing as the conclusion of an introduction and as the major premise of a corresponding elimination. For more on the philosophical consequences of taking the harmony of rules seriously, see Dummett’s classic articles ‘The Justification of Deduction’ (1973) and ‘The Philosophical Basis of Intuitionistic Logic’ (1975), in his collection *Truth and Other Enigmas*, Duckworth, 1978. See also Prawitz’s important paper ‘Meaning and proofs: On the conflict between classical and intuitionistic logic’, *Theoria* 43, 1977, pp. 2–40. Dummett discusses the philosophical implications of Prawitz’s ideas further in *The Logical Basis of Metaphysics*, Harvard University Press, 1991. For an explication of harmony of rules in terms of *logical strength* of conclusions and major premises, see Neil Tennant, *Natural Logic*, Edinburgh University Press, 1978, pp. 74–77.

² See N. Tennant, ‘Perfect validity, entailment and paraconsistency’, *Studia Logica* XLIII, 1984, pp. 179–198; ‘Natural Deduction and Sequent Calculus for Intuitionistic Relevant Logic’, *Journal of Symbolic Logic* 52, 1987, pp. 665–680; *Autologic*, Edinburgh University Press, 1992; and *The Taming of The True*, Clarendon Press, Oxford, 1997, Ch. 10.

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