Harmony in a sequent setting

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Abstract

In response to a problem pointed out by Steinberger (2009), the Harmony Principle of Tennant forthcoming (b) is here given a sequent formulation in order better to illustrate its application to the existential quantifier.

1. The Harmony of the rules for the standard existential quantifier

The usual definition of the notion 'φ is a (logically) stronger proposition than ψ' is that one can deduce ψ from φ, but not conversely.

An alternative definition, which will prove useful for present purposes, is that from any sequent of the form Γ, ψ:θ one can derive the sequent Γ, φ:θ. This stress on the respective roles of φ and of ψ in arguments goes back to §3 of Frege's *Begriffsschrift*, where he defined identity of the *propositional content* of two sentences in terms of their intersubstitutability, *salva validitate*, as premises of arguments.

Definition. Let $A(x)$ be a formula with just the variable x free. Then for any closed term term t, we define $A(t)$ to be the sentence that results from $A(x)$ by replacing all free occurrences of x with occurrences of t. We shall also abbreviate $\exists xA(x)$ to $\exists xA$.

In sequent form, the rule of existential introduction (or '∃ on the right') is

(∃-I) $\dfrac{\Delta:A(t)}{\Delta:\exists xA}$,

and the rule of existential elimination (or '∃ on the left') is

(∃-E) $\dfrac{\Delta, A(a):\psi}{\Delta,\exists xA:\psi}$ where a does not occur in the lower sequent.

Lemma 1. Suppose φ satisfies the conditions on $\exists xA$ in the statement of (∃-I). Then by making full use of (∃-E) – but no use of (∃-I) – we can show that $\exists xA$ is at least as strong as φ, i.e. from any sequent of the form Γ, φ:θ we can derive the sequent Γ, $\exists xA$:θ.

Proof. That φ satisfies the conditions on $\exists xA$ in the statement of (∃-I) means that the following sequent-inference (†) may be used:

$$(\dagger) \qquad \frac{\Delta : A(t)}{\Delta : \varphi}$$

Let *a* be any parameter not occurring in the sequent Γ, ∃*xA*:θ. The following sequent derivation concludes our proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A(a): A(a)}\,\text{(REFLEXIVITY)}}{A(a):\varphi}\,(\dagger) \qquad \Gamma, \varphi:\theta}{\Gamma, A(a):\theta}\,\text{(CUT)}}{\Gamma, \exists xA:\theta}\;\text{full use of } (\exists\text{-E})$$

Note the crucial application of CUT; more on this later.

Lemma 2. Suppose φ satisfies the conditions on ∃*xA* in the statement of (∃-E). Then by making full use of (∃-I) – but no use of (∃-E) – we can show that ∃*xA* is at least as weak as φ, i.e. from any sequent of the form Γ, ∃*xA*:θ we can derive the sequent Γ, φ:θ.

Proof. That φ satisfies the conditions on ∃*xA* in the statement of (∃-E) means that the following sequent-inference (††) may be used:

$$(\dagger\dagger) \qquad \frac{\Delta, A(a):\psi}{\Delta, \varphi:\psi}\ , \text{ where } a \text{ does not occur in the lower sequent.}$$

Let *a* be any parameter not occurring in the sequent Γ, φ:θ. The following sequent derivation concludes our proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A(a): A(a)}\,\text{(REFLEXIVITY)}}{A(a):\exists xA}\,(\exists\text{-I}) \qquad \Gamma, \exists xA:\theta}{\Gamma, A(a):\theta}\,\text{(CUT)}}{\Gamma, \varphi:\theta}\,(\dagger\dagger)$$

Again, note the crucial application of CUT; more on this later.

2. The apparent Harmony of the rules for a deviant quantifier

We now need to check that the foregoing proofs of the Harmony of (∃-I) and (∃-E) cannot be subverted into proofs of the Harmony of (∃-I) and any deviant rule (∃-E') where (∃-E') is like (∃-E) except in so far as it may lack some or all of the restrictions on the parameter *a*. For definiteness in the following discussion, let us consider Steinberger's deviant elimination rule that imposes no restrictions at all on occurrences of the parameter *a*. Also, in order to aid the understanding, let us use E for the would-be 'existential' quantifier that is governed by the (ordinary) introduction rule and the new deviant elimination rule. So, for clarity at the outset, we are considering a quantifier E subject to the following rules:

(E-I) $$\frac{\Delta: A(t)}{\Delta: \mathrm{E}xA}$$

(which is exactly like the old rule (∃-I)) and the new deviant rule

(E-E) $$\frac{\Delta, A(a):\psi}{\Delta, \mathrm{E}xA:\psi}$$, where there are no restrictions on *a*.

Our earlier proofs of Lemmas (1) and (2) would now be re-worked as follows.

Lemma 3. Suppose φ satisfies the conditions on E*xA* in the statement of (E-I). Then by making full use of (E-E) – but no use of (E-I) – we can show that E*xA* is at least as strong as φ, i.e. from any sequent of the form Γ, φ:θ we can derive the sequent Γ, E*xA*:θ.

Proof. That φ satisfies the conditions on E*xA* in the statement of (E-I) means that the following sequent-inference (#) may be used:

(#) $$\frac{\Delta: A(t)}{\Delta:\varphi}$$

Let *a* be any parameter – perhaps even one that occurs in the sequent Γ, E*xA*:θ. The following sequent derivation concludes our proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A(a):A(a)}\,\text{(REFLEXIVITY)}}{A(a):\varphi}\,(\#)\qquad \Gamma, \varphi:\theta}{\Gamma, A(a):\theta}\,\text{(CUT)}}{\Gamma, \mathrm{E}xA:\theta}\,\text{‘full use’ of (E-E)}$$

Lemma 4. Suppose φ satisfies the conditions on E*xA* in the statement of (E-E). Then by making full use of (E-I) – but no use of (E-E) – we can show that E*xA* is at least as weak as φ, i.e. from any sequent of the form Γ, E*xA*:θ we can derive the sequent Γ, φ:θ.

Proof. That φ satisfies the conditions on E*xA* in the statement of (E-E) means that the following sequent-inference (##) may be used:

(##) $\dfrac{\Delta, A(a):\psi}{\Delta, \varphi:\psi}$, where *a* may even occur in the lower sequent.

The following sequent derivation concludes our proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A(a):A(a)}\,(\text{REFLEXIVITY})}{A(a):\mathrm{E}xA}\,(\text{E-I}) \qquad \Gamma, \mathrm{E}xA:\theta}{\Gamma, A(a):\theta}\,(\text{CUT})}{\Gamma, \varphi:\theta}\,(\#\#)$$

3. Discussion

It would appear that these two proofs (of Lemmas (3) and (4)) accomplish exactly the same goal as was accomplished by the two proofs (of similar form) of Lemmas (1) and (2). Therefore, on our account of Harmony, the rule (E-I) is in Harmony with the deviant rule (E-E) – an unacceptable conclusion. But the reasoning to this conclusion is over-hasty.

On the present author's approach to these foundational proof-theoretical matters, the rule of CUT *is not taken as a primitive structural rule*. It is not available as a presupposition in discussions of the field of the deducibility relation. It is not available *tout court* for application to just any sentence, including those with E as their dominant operator. Rather, as Tennant forthcoming (a) is designed to show, the rule of CUT is an *admissible* rule. Its admissibility has to be established *by appeal to the introduction and elimination rules governing the various logical operators*. In the foregoing proofs of Lemmas (1) and (2), we were able to appeal to CUT because its admissibility had been established for the language based on the usual logical operators, including the standard quantifier ∃. In the subsequent proof of Lemma (4), analogous appeal is made to CUT on a sentence whose dominant operator is E. But we have not yet secured CUT as an admissible rule for a language containing E.

In Tennant forthcoming (a), crucial use is made of the reduction procedures for the logical operators, when showing that CUT is admissible for the language based on those operators. So the question to be faced here is whether we can furnish a reduction procedure for the deviant quantifier E.

The answer is negative. In Tennant forthcoming (a), the admissibility of CUT is established by the following theorem.

Theorem [Cut Elimination for Core Proof]
There is an effective method [,] that transforms any two core proofs

$$\begin{array}{cc} \Delta & A,\Gamma \\ \Pi & \Sigma \\ A & \theta \end{array}$$

into a core proof [Π,Σ] of θ or of ⊥ from (some subset of) Δ∪Γ.

The operation [,] is defined inductively on the complexity of the proofs Π and Σ, and the complexity of the cut-sentence *A*.

One of the cases in the inductive step is disposed of by the following reduction procedure for the standard existential quantifier ∃. We assume that the proof Σ features the parameter *a* for existential elimination. By Σ[*a/t*] (abbreviated to Σ(*t*)) we mean the result of replacing all relevant parametric occurrences of *a* within Σ by occurrences of *t*.

$$
\left[\begin{array}{c} \Xi \\ \Theta \\ \dfrac{A(t)}{\exists x A} \end{array} \; , \; \dfrac{\exists x A \qquad \begin{array}{c} \overset{(i)}{A(a), \Gamma} \\ \Sigma \\ \theta \end{array}}{\theta}{\scriptstyle (i)} \right] \quad = \quad \left[\begin{array}{c} \Xi \\ \Theta \\ A(t) \end{array} \; , \; \begin{array}{c} A(t), \Gamma \\ \Sigma(t) \\ \theta \end{array} \right]
$$

Note that in the reduct on the right, the proof Σ(*t*) is obtained from the subproof Σ for (∃-E) by substituting occurrences of the term *t* for the parametric occurrences of *a* within Σ. So Σ(*t*) is actually

$$
\left(\begin{array}{c} A(a), \Gamma \\ \Sigma \\ \theta \end{array}\right)[a/t]
$$

In order for the substitution of *t* for parametric occurrences of *a* to yield *A*(*t*) as an undischarged assumption of Σ[*a/t*], the formula *A* must be prohibited from containing any occurrences of *a*. (We need *A*(*t*) to be an undischarged assumption of Σ[*a/t*] in order for that assumption-occurrence of *A*(*t*) to match the conclusion-occurrence of *A*(*t*) in Θ. If they did not match, the operation [,] would not be able to apply recursively.) It is also crucial, for the proof of the sought result, that Γ and θ remain undisturbed by such substitutions. But that in turn means that the parameter *a* should not be allowed to occur in Γ or in θ. The upshot of these syntactic considerations is that the parameter *a* should not be allowed to occur within *A*, within (any member of) Γ or within θ. Violate any of these requirements, and the reduction procedure for ∃ will not be applicable.

But the would-be quantifier E is supposed to violate these requirements. We have not yet shown that there is *no* reduction procedure for E. But we have shown that *if* there is such a reduction procedure, then it cannot have anything like the shape of the reduction procedure for ∃.

How, then, might one show conclusively that no reduction procedure for E is possible? The best way would be to suppose, for *reductio*, that CUT on E-sentences is admissible, and then to show that this would enable one to prove certain sequents that one knows, in advance, are not (logically) provable. And the best sequent to choose for this purpose would be an atomic sequent – one involving only atomic sentences. For, since it would involve no logical operators, it would not be logically provable.

We are arguing, in effect, for a certain *conservativeness requirement*:

> The rules for a logical operator should not allow one to prove (on the assumption that CUT is admissible) any atomic sequent.

The following proof serves our stated purposes:

$$\dfrac{\dfrac{F(t):F(t)}{F(t):\mathrm{E}xF(x)}\,\text{(E-I)} \qquad \dfrac{F(a):F(a)}{\mathrm{E}xF(x):F(a)}\,\text{(E-E)}}{F(t):F(a)}\,\text{(CUT)}$$

We may conclude, therefore, that there is no reduction procedure for the rules for (E-E). As a consequence, CUT cannot hold for sentences with E dominant, and we are unable to prove (by the method used above) that the rules for E are in Harmony.

These reflections accord a certain priority to an operator's being possessed of a reduction procedure – or, equivalently, to its rules' obeying the conservativeness requirement. It is that which makes CUT admissible on sentences with that operator dominant. That, in turn, enables one to prove (by the method used above) that the rules for the operator are in harmony.

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