On Maintaining Concentration

NEIL TENNANT

Peter Milne (this issue, [1]) has produced a neat counterexample to my conjecture in [2] that every sequent provable in the system IR of intuitionistic relevant logic is a substitution instance of an intuitionistically valid sequent that has no intuitionistically valid proper subsequent. While the Popperian in me rejoices, the relevantist/intuitionist in me might have to resign itself with disappointment to the prospect that the conjecture might not hold for any system worthy of the title of intuitionistic relevant logic. Now while I do not think the truth of the conjecture is essential to IR, it would still be welcome if it were so. What, then, can be done to save the situation in the light of Milne's counterexample?

The sequent that Milne provides as his counterexample is A v B, A, B : A & B and its proof in IR is as follows:[1]

$$(\Theta) \qquad \frac{A \vee B \qquad \dfrac{\overline{A}^{(1)} \quad B}{A \,\&\, B} \qquad \dfrac{A \quad \overline{B}^{(1)}}{A \,\&\, B}}{A \,\&\, B}\,{\scriptstyle (1)}$$

Let us recall the guiding idea behind the treatment of relevance embodied in the systems CR and IR. The main aim was to avoid thinning, or dilution – structural inferences (in a sequent system) that become sources of irrelevant connections between sets of premisses and conclusions drawn from them. The slogan here could be:

(γ) Maintain Concentration!

What Milne has shown is that concentration lapsed when I formulated the rule of v-Elimination (proof by cases) for IR.

Ironically, the implemented proof-finding algorithm for propositional IR developed in *Autologic* manages to maintain concentration in a way that would make the little proof Θ above impossible for the algorithm to 'discover'.[2] What has been overlooked in the proof theory, but not in the proof-finding algorithm, is the need to impose certain explicit conditions on the rule of v-Elimination in order to maintain concentration. These

[1] For the rules of IR for ¬, v and & see [3]; for the rules for the full system with ⊃ as well, see [4].

[2] Dov Gabbay was the first to suggest that the specification of a proof-finding algorithm might be taken as part of the overall specification of the very identity of the logical system in question. The interesting philosophical implications of this proposal deserve discussion in a separate paper.

ANALYSIS 54.3, July 1994, pp. 143–152. © Neil Tennant

extra conditions do no damage to the 'completeness' of IR *vis-à-vis* its parent system I of ordinary intuitionistic logic; and they ensure that unnecessarily complex proofs are not served up, and that Milne's counterexample is obviated.

But how, one might ask, can one avoid Milne's counterexample and still maintain that the relevant system IR is 'complete' *vis-à-vis* the parent system I of ordinary intuitionistic logic? The main goal in the quest for 'completeness' was that a theorem of the following form should hold, for the system IR that one finally favoured:

> *Theorem*[3]: If A can be deduced in I from the set X of assumptions, then either A or ⊥ (absurdity) can be deduced in IR from some subset of X

This theorem guarantees *epistemic gain* from the relevantizing effect of IR. It means that nothing, from the intuitionist's point of view, is lost. In particular, every intuitionistically inconsistent set is provably inconsistent in IR; every intuitionistic theorem is a theorem of IR; and every intuitionistic consequence of an intuitionistically consistent set of assumptions can be deduced from (some subset of) that set of assumptions in IR. So long as the theorem just stated holds for the system IR, we can say 'good riddance' to anything we 'lose' in the move from I to IR.

I shall now repair the lapse of concentration in my earlier formulation of the proof system I gave for IR, by supplying some obvious and sensible precautions against irrelevance when applying one's inference rules. The tightening of the allowable connections between sets of premisses, and conclusions drawn from them, will not impair the main theorem. The new system IR resulting from the extra conditions to be imposed is one for which that theorem holds. The extra conditions, as already mentioned, are ones that it is prudent to adopt in automated proof-search, where a deductive problem of the form X?- A is solved as soon as one has produced a proof either of A or of ⊥ from any subset of X.

The idea is really very simple:

> (γ_1) In any application of a rule of inference R to immediate subproofs $\Pi_1, \ldots, \Pi_n$ to obtain a proof of A from X, we must ensure that the set of undischarged assumptions of any of the subproofs Π_i whose conclusion happens to be either A or ⊥ is not a subset of X.

For, if some such subproof Π_i had either A or ⊥ as its conclusion, and had its set of premisses Δ_i included in X, then Π_i itself would be a solution to the deductive problem X?- A. It would accordingly be both unnecessary

[3] I first proved this theorem for full first-order logic in 1987. The proof is in [5].

and prolix to seek to apply the rule of inference R to 'build up' a proof of A from X. The effort would be wasted, and the deductive insight afforded by Π_i would be obscured.

Let this condition (γ_1) now be imposed uniformly on all applications of rules of inference in IR – just as we impose uniformly the condition that proofs in IR should be in normal form. It is clear that this is *not* an *ad hoc* move. So far I have said nothing addressed to Milne's counterexample in particular. My new and explicit proof-theoretical condition is motivated solely by the desire to avoid, by pretty obvious means, the effects of dilution, within proofs, of results 'already proved' in the earlier innards of those proofs. The condition has already been incorporated in the design of efficient proof-search algorithms, so one may as well incorporate it into the proof theory too.

When we look at the effect of our new requirement (γ_1) on the proof Θ given above of Milne's counterexample, we see that the final step of Θ would no longer count as correct. For, this final step of $\vee$-Elimination is proposed in order to obtain a proof of A & B 'from' A, B, A $\vee$ B. Before we can take that final step, however, our new condition (γ_1) enjoins us to check whether any of the subordinate proofs do not already establish that result – that is, A, B, A $\vee$ B : A & B – or, indeed, a result that would be even better – namely, any subsequent of A, B, A $\vee$ B : A & B. (Here by a subsequent we mean any sequent whose premisses are among A, B, A $\vee$ B and whose conclusion is either A & B or $\perp$.)

But it turns out that *both* of the subordinate proofs in question (which happen to be identical) establish the subsequent A, B : A & B of the 'overall' sequent A, B, A $\vee$ B : A & B, which it would be the supposed achievement of the contemplated final application of the rule of $\vee$-Elimination to have established. We can therefore forswear the extended 'proof' with a clear conscience; we can maintain concentration.

Will this new requirement (γ_1), however, serve to rule out *every* Milne-type example? I think that it can be made to, with a little extra care. (γ_1) needs to go hand-in-hand with two further conditions (γ_2) and (γ_3), concerning the introduction and elimination rules for conjunction, which I shall now motivate before formulating them below. Consider the proof

(Ξ)

$$\dfrac{A \vee B \qquad \dfrac{\dfrac{}{A}{}^{(1)} \quad \dfrac{B \,\&\, A}{B}}{A \,\&\, B} \qquad \dfrac{\dfrac{B \,\&\, A}{A} \quad \dfrac{}{B}{}^{(1)}}{A \,\&\, B}}{A \,\&\, B}\,(1)$$

If we imagine ourselves working down from the top within Ξ, reaching the point at which we have the two subproofs for the final step of $\vee$-Elimination, we find that we have, it seems, satisfied the new condition (γ_1) to be imposed on that final step. For the set {A, B & A} of undischarged assumptions in the first case-proof is not a subset of the set {A v B, B & A} of overall assumptions of the proof Ξ that would result from that final step; nor is the set {B, B & A} such a subset. So, it would seem, this proof Ξ stands as correct. But would that not be an embarrassment? For the sequent A v B, B & A : A & B is as much a counterexample to my conjecture as was Milne's sequent A, B, A v B : A & B, and for the same reasons as Milne advanced.

But wait! I said above 'If we imagine ourselves working down from the top ...'; and this is not exactly what we do when we search for proofs, guided by considerations of relevance. What we do, rather, is (in natural deduction terms) work *both* upwards *and* downwards. In *Autologic* I developed a version of natural deduction that would make this feature of proof-search more transparent. I called the new system 'hybrid' because it combines the virtues of the natural deduction system with those of the sequent system, and with none of their respective drawbacks. In the hybrid system major premisses for eliminations stand proud at tops of branches, with no sentences 'above' them. The hybrid system has its rules of &-Elimination and ⊃-Elimination, like the usual rule of v-Elimination, stated in a 'parallel' form rather than their conventional 'serial' form:

(&-E)

$$\dfrac{A \,\&\, B \qquad \begin{array}{c} \overline{\;\;}^{(i)}\square\overline{\;\;}^{(i)} \\ A\;,\;B \\ \vdots \\ C \end{array}}{C}\,(i)$$

Here the discharge notation means that at least one of A or B must have been used as assumptions within the subordinate proof of C; and the application of the rule discharges all assumption occurrences of A and of B within that subordinate proof.

(⊃-E)

$$\dfrac{A \supset B \qquad \begin{array}{c} \\ \\ \vdots \\ A \end{array} \qquad \begin{array}{c} \square\overline{\;\;}^{(i)} \\ B \\ \vdots \\ C \end{array}}{C}\,(i)$$

Here the discharge notation means that B must have been used as an assumptions within the 'major' subordinate proof; and the application of the rule discharges all such assumption occurrences of B.

Let us now see the effect of re-framing the rule of &-E this way. It might at first seem disappointing, and appear not to achieve anything for us in the way of blocks to Milne-type examples. For here is a re-formulation of the proof Ξ most recently given, using the new form of &-E:

(Π)

$$\dfrac{A \vee B \qquad \dfrac{{}^{(3)}\overline{A} \qquad \dfrac{B \,\&\, A \quad \overline{B}^{(1)}}{B}{\scriptstyle (1)}}{A \,\&\, B} \qquad \dfrac{\dfrac{B \,\&\, A \quad \overline{A}^{(2)}}{A}{\scriptstyle (2)} \qquad \overline{B}^{(3)}}{A \,\&\, B}}{A \,\&\, B}{\scriptstyle (3)}$$

But, as the designer of automated proof search will be quick to point out, this proof Π is unnecessarily prolix, and should not be aimed at by an efficient proof-finding algorithm. One should, rather, take advantage of the new power afforded by the parallel form of &-E to discharge assumptions 'at one stroke'. A proof that does this is the following:

(Σ)

$$\dfrac{B \,\&\, A \qquad \dfrac{A \vee B \qquad \dfrac{\overline{A}^{(1)} \quad \overline{B}^{(2)}}{A \,\&\, B} \qquad \dfrac{\overline{A}^{(2)} \quad \overline{B}^{(1)}}{A \,\&\, B}}{A \,\&\, B}{\scriptstyle (1)}}{A \,\&\, B}{\scriptstyle (2)}$$

Note, however, that the step of v-Elimination in this 'proof' Σ would violate the new condition (γ_1) that I have imposed in the interest of maintaining concentration. On finding thus that it is impossible to proceed via v-Elimination, one would try instead to prove A & B at its upper occurrence by &-Introduction from the other available premisses, namely A, B. This one can do in one step, so the resulting proof would be:

(Σ_1)

$$\dfrac{B \,\&\, A \qquad \dfrac{\overline{A}^{(1)} \quad \overline{B}^{(1)}}{A \,\&\, B}}{A \,\&\, B}{\scriptstyle (1)}$$

Have we reached an impasse? We appear to have, in the new system of IR, one proof (Π) of A v B, B & A : A & B that goes through without violating (γ_1), and one attempted 'proof' (Σ) that doesn't go through, because it does violate (γ_1). What has been achieved?

My reply is that in the interests of maintaining concentration we are still at liberty to impose any structural conditions we like on proofs (that is, on the application of rules of inference) – provided only that these conditions are decidable, and that imposing them does not confute the Theorem stated above.

Now I want to rule out Π as well as Σ. So a new structural condition that I would like to impose – one which merely registers, in the proof theory, the insights involved in computational proof-search – is the following:

(γ_2) Applications of &-E should be 'as low down as they can possibly be, but no further down than they ought to be'.

Note that &-E is a 'safe' rule. That is, a positive solution to the deductive problem [... A & B ...] ?- C is available if and only if a positive solution is available to the deductive sub-problem [... A, B ...] ?- C. So the procedural, proof-search way of stating the condition is as follows:

(γ_2) When given any deductive problem with a conjunction among the premisses, if one decides to apply an elimination rule at that stage, one should reduce the deductive problem immediately to the subproblem that has the two conjuncts in place of that conjunction. (If there are more than two conjunctive premisses, reduce them in any order.)

The new condition (γ_2) can be thought of as helping to specify a more stringent notion of normal form. In ordinary intuitionistic logic, of course, restriction of proofs to these more stringent normal forms involves no loss of completeness. For every proof in ordinary normal form can be re-cast so as to have all its steps of &-E shuffled down as far as possible. A step of &-E will shuffle down past any rule-application that does not discharge the major premiss of that step of &-E.

When, however, we come to *relevantize* so as to obtain a satisfactory version of IR, this extra ingredient (γ_2) in the specification of normal form becomes useful in maintaining concentration. The effect will be achieved by restricting applications of rules of inference in such a way as to prevent applications of &-E from occurring 'too high up' – or, correspondingly, 'too late' during bottom-up search in the hybrid system. So another way of stating our new condition is as follows:

(γ_2) In the usual inductive definition of proof, we will require that no rule R may be applied to subproofs ending with an application of &-E, unless the major premiss of that application of &-E is discharged by the contemplated application of R

The following proof shows that something extra is still required in our effort to maintain concentration. I owe this counterexample to Peter Milne (private correspondence) :

(Ω)

$$\dfrac{A \vee B \qquad \dfrac{\dfrac{\dfrac{\overline{A}^{(3)} \quad \overline{\neg A}^{(1)}}{\bot}}{\neg\neg A}{\scriptstyle (1)} \qquad \neg\neg B}{\neg\neg A \,\&\, \neg\neg B} \qquad \dfrac{\neg\neg A \qquad \dfrac{\dfrac{\overline{B}^{(3)} \quad \overline{\neg B}^{(2)}}{\bot}}{\neg\neg B}{\scriptstyle (2)}}{\neg\neg A \,\&\, \neg\neg B}}{\neg\neg A \,\&\, \neg\neg B}{\scriptstyle (3)}$$

The sequent hereby proved – $A \vee B, \neg\neg A, \neg\neg B : \neg\neg A \,\&\, \neg\neg B$ – is not a substitution instance of a perfectly valid sequent. But at no stage does the proof Ω violate (γ_1) or (γ_2).

There is a simple way, however, to show how the illicit dilution within the proof Ω has been effected. Simply apply to Ω the transformation that guarantees that any proof ending with a step of $\vee$-E applied to subproofs at least one of which ends with &-I can be turned into a proof ending with &-I whose subproofs end with ∨-E:

$$\dfrac{A \vee B \qquad \dfrac{\begin{matrix}\overline{A}^{(i)} & \overline{A}^{(i)} \\ \Pi_1 & \Sigma_1 \\ C & D\end{matrix}}{C\&D} \qquad \dfrac{\begin{matrix}\overline{B}^{(i)} & \overline{B}^{(i)} \\ \Pi_2 & \Sigma_2 \\ C & D\end{matrix}}{C\&D}}{C\&D}{\scriptstyle (i)} \quad \Rightarrow \quad \dfrac{\dfrac{A \vee B \quad \begin{matrix}\overline{A}^{(i)} \\ \Pi_1 \\ C\end{matrix} \quad \begin{matrix}\overline{B}^{(i)} \\ \Pi_2 \\ C\end{matrix}}{C}{\scriptstyle (i)} \qquad \dfrac{A \vee B \quad \begin{matrix}\overline{A}^{(j)} \\ \Sigma_1 \\ D\end{matrix} \quad \begin{matrix}\overline{B}^{(j)} \\ \Sigma_2 \\ D\end{matrix}}{D}{\scriptstyle (j)}}{C\&D}$$

(The reader can easily check that the transformation is also available when one of the case-proofs of C & D is replaced by a proof of absurdity.) The result of applying this transformation to the Milne-proof Ω is:

$$\dfrac{\dfrac{A \vee B \qquad \dfrac{\dfrac{\overline{A}^{(i)} \quad \overline{\neg A}^{(1)}}{\bot}}{\neg\neg A}{\scriptstyle (1)} \qquad \neg\neg A}{\neg\neg A}{\scriptstyle (i)} \qquad \dfrac{A \vee B \qquad \neg\neg B \qquad \dfrac{\dfrac{\overline{B}^{(j)} \quad \overline{\neg B}^{(2)}}{\bot}}{\neg\neg B}{\scriptstyle (2)}}{\neg\neg B}{\scriptstyle (j)}}{\neg\neg A \& \neg\neg B}$$

in which it is now conspicuous that the applications of ∨-E are illicit. For they violate the requirement that the relevant case assumption should actually have been used, and be available for discharge, within each case proof. In the leftmost step of ∨-E the second 'case proof' is just ¬ ¬ A, with no use made of the second case assumption B. Likewise in the rightmost step of ∨-E the first 'case proof' is just ¬ ¬ B, with no use made of the first case assumption A. This shows that there was something rotten – something over-diluted – in the original Milne-proof Ω. The reader will by now no doubt have anticipated my next condition on proofs:

(γ_3) Applications of &-I should be 'as low down as they can possibly be'.

Note that &-I is a 'safe' rule. That is, a positive solution to the deductive problem X ?- A & B is available if and only if positive solutions are available to both of the deductive sub-problems X ?-A and X ?-B. So the procedural, proof-search way of stating the condition is as follows:

(γ_3) Given a deductive problem X ?- A & B, break it down immediately into the deductive sub-problems X ?-A and X ?-B. After success on both of these, apply &-I for success on the original problem X ?- A & B. But if one fails on either of these, one fails on the original problem too.

Another way of formulating this requirement within proof theory is as follows:

(γ_3) In the usual inductive definition of proof, we will require that ∨-E may not be applied to subproofs ending with an application of

&-I, and that applications of ⊃-E may not be applied with a major subproof that ends with an application of &-I

In cases where we have a conjunctive conclusion and at least one conjunction among the premisses, (γ_2) is to have precedence over (γ_3). That is, we shall deal with the conjunctive premiss first. The combined force of (γ_2) and (γ_3) is clear: when searching for proofs, deal with all dominant occurrences of & in the premisses and the conclusion *first*, before dealing with any other logical operators.

One can see how the procedural restriction (γ_3) would make it impossible to discover the Milne-type proof just given. Starting with the deductive problem [A v B, ¬ ¬ A, ¬ ¬ B] ?- ¬ ¬ A & ¬ ¬ B we would, by (γ_3), split it up into the two sub-problems [A v B, ¬ ¬ A, ¬ ¬ B] ?- ¬ ¬ A and [A v B, ¬ ¬ A, ¬ ¬ B] ?- ¬ ¬ B. These would each fall immediately to the observation that the desired conclusion was available as a premiss. The unnecessary premiss A v B would simply not be used.

Note that (γ_3) by itself would have dealt with the very first of Milne's troublesome 'proofs', the one called Θ above. The deductive problem there was [A v B, A, B] ?- A & B. If we heed (γ_3) then this reduces immediately to the two sub-problems [A v B, A, B] ?- A and [A v B, A, B] ?- B. Each falls to the observation that the conclusion is available as a premiss. Thus the proof consists in a single step of &-I, and the premiss A v B proves to be irrelevant.

Indeed, it is instructive to look once again at all three of the closet diluters that have been causing the problems. They are:

(Θ)

$$\dfrac{A \vee B \qquad \dfrac{\overline{A}^{(1)} \quad B}{A \,\&\, B} \qquad \dfrac{A \quad \overline{B}^{(1)}}{A \,\&\, B}}{A \,\&\, B}$$

(Ξ)

$$\dfrac{A \vee B \qquad \dfrac{\overline{A}^{(1)} \quad \dfrac{B \,\&\, A}{B}}{A \,\&\, B} \qquad \dfrac{\dfrac{B \,\&\, A}{A} \quad \overline{B}^{(1)}}{A \,\&\, B}}{A \,\&\, B}{\scriptstyle (1)}$$

(Ω)

$$\dfrac{A \vee B \qquad \dfrac{\dfrac{\dfrac{\overline{A}^{(3)} \quad \overline{\neg A}^{(1)}}{\bot}}{\neg\neg A}{\scriptstyle (1)} \quad \neg\neg B}{\neg\neg A \,\&\, \neg\neg B} \qquad \dfrac{\neg\neg A \quad \dfrac{\dfrac{\overline{B}^{(3)} \quad \overline{\neg B}^{(2)}}{\bot}}{\neg\neg B}{\scriptstyle (2)}}{\neg\neg A \,\&\, \neg\neg B}}{\neg\neg A \,\&\, \neg\neg B}$$

Note how each of these 'proofs' has steps of &-I occurring above a step of v-E. So (γ_3) alone is an effective block against all of them. It will do no harm, however, to have both (γ_1) and (γ_2) up our sleeves as well.

Our response so far to the three proofs above has been to say 'Aha! they all have &-I occurring too low down.' Another would be to say 'Ah yes - but, by the same token, they all have v-E occurring too high up.' Limitations of space prevent me from writing out in full here all the transformations on proofs that suffice to warrant my next claim, so I shall simply assert it, leaving its (renewed) verification to the interested reader as an exercise. The claim is that applications of v-E in IR proofs can always be shuffled *upwards* in the following circumstances:

1. one of the minor case-proofs ends with absurdity, and the other ends with an introduction;
2. both of the minor case-proofs end with ¬-I, ⊃-I, or &-I;
3. both of the minor case-proofs end with v-I 'on the right' (e.g. getting C v D from C); or
4. both of the minor case-proofs end with v-I 'on the left' (e.g. getting C v D from D)

The *only* situation where v-E can *not* be shuffled upwards past terminal introductions in its minors is where one minor ends with v-I 'on the right' and the other minor ends with v-I 'on the left'. Such is the case when proving C v D from D v C, for example.

These observations motivate the following condition:

(γ_4) Applications of v-E should be 'as high up as they can possibly be'.

That is to say: we should make their conclusions as nearly atomic as is possible. (γ_4) rules out the following modification Ω_0 of Ω (due again to Peter Milne, private correspondence), which would escape the strictures of (γ_1), (γ_2) and (γ_3):

(Ω_0)

$$\dfrac{A \vee B \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overline{A}^{(3)} \quad \overline{\neg A}^{(1)}}{\bot}}{\neg\neg A}{\scriptstyle (1)} \qquad \neg\neg B}{\neg\neg A \,\&\, \neg\neg B}}{C \vee (\neg\neg A \,\&\, \neg\neg B)} \qquad \dfrac{\dfrac{\neg\neg A \qquad \dfrac{\dfrac{\overline{B}^{(3)} \quad \overline{\neg B}^{(2)}}{\bot}}{\neg\neg B}{\scriptstyle (2)}}{\neg\neg A \,\&\, \neg\neg B}}{C \vee (\neg\neg A \,\&\, \neg\neg B)}}{C \vee (\neg\neg A \,\&\, \neg\neg B)}$$

(γ_4) would oblige us to shuffle the terminal step of v-E up past the steps of v-I in the case proofs. Then (γ_4) (and indeed (γ_3)) would oblige us to shuffle the step of v-E further up again, past the steps of &-I. By that stage the resulting applications of v-E would be illicit, for they would (as we saw earlier with the transform of Ω) violate the requirement that the relevant case assumption should actually have been used, and be available for discharge, within each case proof. So, since obligatory transformations on the would-be proof Ω_0 turn it into something ill-formed, Ω_0 itself is ill-formed.

I propose therefore that all the rules of IR should be just as laid out in *Autologic*, only subject to our further provisos (γ_1), (γ_2), (γ_3) and (γ_4). Call the new system IR[γ]. Can we now rest assured that Milne-type counterexamples cannot arise for IR[γ]? I hesitate to say so. The Popperian within me conjectures thus; while the relevantist/intuitionist within me stands ready to seek new conditions (γ_i) that will avoid dilution, yet preserve our Theorem, if new counterexamples to this new conjecture are produced to show that such conditions will be required. In the absence of a proof of the conjecture, that is the best one can do. It is always a battle to maintain concentration.

The Ohio State University
Columbus, OH 43210-1365, USA
ntennant@magnus.acs.ohio-state.edu
&
Churchill College, Cambridge
nwt1000@cus.cam.ac.uk

References

[1] Peter Milne, 'Intuitionistic Relevant Logic and Perfect Validity', *Analysis* 54 (1994) 140–2.
[2] Neil Tennant, 'Perfect Validity, Entailment and Paraconsistency', *Studia Logica* 43 (1984) 179–98.
[3] Neil Tennant, *Anti-Realism and Logic* (Oxford: Clarendon Press, 1987).
[4] Neil Tennant, *Autologic* (Edinburgh: Edinburgh University Press, 1992).
[5] Neil Tennant, 'Intuitionistic Mathematics does not need Ex Falso Quodlibet', *Topoi*, forthcoming.