The Satisfiability Theorem

by

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Theorem 1 *Suppose Δ is consistent and decisive. Then for every sentence φ:*
(i) if $\Delta \vdash \varphi$, then $|\Delta| \Vdash \varphi$; and
(ii) if $\Delta, \varphi \vdash \bot$, then $|\Delta|, \varphi \Vdash \bot$.

Proof. By induction on φ.
Basis. Suppose φ is atomic. For (i), suppose $\Delta \vdash \varphi$. Then $\varphi \in |\Delta|$, whence by (λ) we have $|\Delta| \Vdash \varphi$. For (ii), suppose $\Delta, \varphi \vdash \bot$. By $(\vdash \neg)$ we have $\Delta \vdash \neg\varphi$. But φ is atomic. So $(\neg\varphi) \in |\Delta|$. By rule (α) we have $|\Delta|, \varphi \Vdash \bot$.

Inductive Hypothesis (IH). Suppose (i) and (ii) hold for ψ and for θ.

Inductive Step. By cases, according as φ is of the form (1) $\neg\psi$, (2) $\psi \wedge \theta$, (3) $\psi \vee \theta$ or (4) $\psi \to \theta$.

Case 1. For (i), suppose $\Delta \vdash \neg\psi$. By (Ref) and $(\neg \vdash)$ we have $\psi, \neg\psi \vdash \bot$. Hence by CUT FOR ABSURDITY we have $\Delta, \psi \vdash \bot$. By IH(ii) we have $|\Delta|, \psi \Vdash \bot$. By $(V\neg)$ we have $|\Delta| \Vdash \neg\psi$, as required.

For (ii), suppose $\Delta, \neg\psi \vdash \bot$. Suppose for *reductio* that $\Delta, \psi \vdash \bot$. By $(\vdash \neg)$ we would have $\Delta \vdash \neg\psi$. Hence by CUT FOR ABSURDITY we would

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have $\Delta \vdash \bot$, contradicting the consistency of Δ. Since Δ is decisive, we conclude $\Delta \vdash \psi$, and discharge our *reductio* assumption. By IH(i), $|\Delta| \Vdash \psi$. By $(F\neg)$ we have $|\Delta|, \neg\psi \Vdash \bot$, as required.

Case 2. For (i), suppose $\Delta \vdash \psi \wedge \theta$. Suppose for *reductio* that $\Delta, \psi \vdash \bot$. By $(\wedge \vdash)$ we would have $\Delta, \psi \wedge \theta \vdash \bot$. By CUT FOR ABSURDITY it would follow that $\Delta \vdash \bot$, contradicting the consistency of Δ. Since Δ is decisive, we conclude $\Delta \vdash \psi$, and discharge our *reductio* assumption. A similar argument shows that $\Delta \vdash \theta$. By IH(i), it now follows that $|\Delta| \Vdash \psi$ and $|\Delta| \Vdash \theta$. Hence by $(V\wedge)$ we have $|\Delta| \Vdash \psi \wedge \theta$, as required.

For (ii), suppose $\Delta, \psi \wedge \theta \vdash \bot$. We seek to show $|\Delta|, \psi \wedge \theta \Vdash \bot$. Since Δ is decisive, either (a) $\Delta \vdash \theta$ or (b) $\Delta, \theta \vdash \bot$.

Case (a): Suppose $\Delta \vdash \theta$. Suppose further for *reductio* that $\Delta \vdash \psi$. By $(\vdash \wedge)$ we would have $\Delta \vdash \psi \wedge \theta$. Hence by CUT FOR ABSURDITY we would have $\Delta \vdash \bot$, contradicting the consistency of Δ. Since Δ is decisive, we have $\Delta, \psi \vdash \bot$. By IH(ii) we have $|\Delta|, \psi \Vdash \bot$. By $(F\wedge)$ we have $|\Delta|, \psi \wedge \theta \Vdash \bot$.

Case (b): Suppose $\Delta, \theta \vdash \bot$. By IH(ii) we have $|\Delta|, \theta \Vdash \bot$. By $(F\wedge)$ we have $|\Delta|, \psi \wedge \theta \Vdash \bot$.

Case 3. For (i), suppose $\Delta \vdash \psi \vee \theta$. We seek to show $|\Delta| \Vdash \psi \vee \theta$. Since Δ is decisive, either (a) $\Delta \vdash \psi$ or (b) $\Delta, \psi \vdash \bot$.

Case (a). Suppose $\Delta \vdash \psi$. By IH(i), $|\Delta| \Vdash \psi$. By $(V\vee)$, $|\Delta| \Vdash \psi \vee \theta$.

Case (b). Suppose $\Delta, \psi \vdash \bot$. Since Δ is decisive, either (a′) $\Delta \vdash \theta$ or (b′) $\Delta, \theta \vdash \bot$.

Case (a′): Suppose $\Delta \vdash \theta$. By IH(i), $|\Delta| \Vdash \theta$. By $(V\vee)$, $|\Delta| \Vdash \psi \vee \theta$.

Case (b′): $\Delta, \theta \vdash \bot$. Note that we are still considering Case (b), in which $\Delta, \psi \vdash \bot$. By $(\vee\vdash)$ we would therefore have $\Delta, \psi \vee \theta \vdash \bot$. Recall that by our main supposition for (i) we have $\Delta \vdash \psi \vee \theta$. By CUT FOR ABSURDITY it would then follow that $\Delta \vdash \bot$, contradicting the consistency of Δ. So this subcase is impossible.

For (ii), suppose $\Delta, \psi \vee \theta \vdash \bot$. Suppose for *reductio* that $\Delta \vdash \psi$. By $(\vdash \vee)$ we would have $\Delta \vdash \psi \vee \theta$. By CUT FOR ABSURDITY we would have $\Delta \vdash \bot$, contradicting the consistency of Δ. Since Δ is decisive, we conclude $\Delta, \psi \vdash \bot$, and discharge our *reductio* assumption. A similar argument shows that $\Delta, \theta \vdash \bot$. By IH(ii) we have $|\Delta|, \psi \Vdash \bot$ and $|\Delta|, \theta \Vdash \bot$. By $(F\vee)$ we have $|\Delta|, \psi \vee \theta \Vdash \bot$.

Case 4. For (i), suppose $\Delta \vdash \psi \to \theta$. We seek to show that $|\Delta| \Vdash \psi \to \theta$.

Since Δ is decisive, either (a) $\Delta \vdash \theta$ or (b) $\Delta, \theta \vdash \bot$.

Case (a): Suppose $\Delta \vdash \theta$. By IH(i) we have $|\Delta| \Vdash \theta$. Hence by $(V \rightarrow)$ we have $|\Delta| \Vdash \psi \rightarrow \theta$.

Case (b): Suppose $\Delta, \theta \vdash \bot$. Suppose for *reductio* that $\Delta \vdash \psi$. By $(\rightarrow\vdash)$ we would have $\Delta, \psi \rightarrow \theta \vdash \bot$. Now recall our main supposition for (i). By CUT FOR ABSURDITY, we would have $\Delta \vdash \bot$, contradicting the consistency of Δ. Since Δ is decisive, we conclude that $\Delta, \psi \vdash \bot$, and discharge our *reductio* assumption. By IH(ii) we have $|\Delta|, \psi \Vdash \bot$. Hence by $(V \rightarrow)$ we have $|\Delta| \Vdash \psi \rightarrow \theta$.

For (ii), suppose $\Delta, \psi \rightarrow \theta \vdash \bot$. We seek to show that $|\Delta|, \psi \rightarrow \theta \Vdash \bot$. Suppose for *reductio* that $\Delta, \psi \vdash \bot$. By $(\vdash\rightarrow)$ we would have $\Delta \vdash \psi \rightarrow \theta$. By CUT FOR ABSURDITY we would have $\Delta \vdash \bot$, contradicting the consistency of Δ. Since Δ is decisive, we conclude $\Delta \vdash \psi$, and discharge our *reductio* assumption. By IH(i) we have $|\Delta| \Vdash \psi$. Now suppose for *reductio* that $\Delta \vdash \theta$. By $(\vdash\rightarrow)$ we would have $\Delta \vdash \psi \rightarrow \theta$. By CUT FOR ABSURDITY, we would have $\Delta \vdash \bot$, contradicting the consistency of Δ. Since Δ is decisive, we conclude $\Delta, \theta \vdash \bot$, and discharge our *reductio* assumption. By IH(ii) we have $|\Delta|, \theta \Vdash \bot$. Hence by $(F \rightarrow)$ we have $|\Delta|, \psi \rightarrow \theta \Vdash \bot$.

Summary of Cases

The foregoing reasoning in the inductive step for results (i) and (ii), according to the possible forms of complex φ ($= \neg\psi,\ \psi \wedge \theta,\ \psi \vee \theta,$ or $\psi \rightarrow \theta$ resp.), can be tabulated in an illuminating fashion. The background assumption is that $\Delta \nvdash \bot$. Moreover, since by further assumption Δ decides each of the two subsentences ψ and θ in just one of two ways (positively or negatively), there are four subcases to consider, as given by the heading of the four columns in the picture below.

The picture contains two sets of four rows, corresponding to the four logical operators. The first set of four rows involves exploration of the assumption (for (i)) that $\Delta \vdash \varphi$. The second set of four rows involves exploration of the assumption (for (ii)) that $\Delta, \varphi \vdash \bot$. We shall call the cell where a given column intersects a given row a 'basic case'.

First we appeal to certain simple deducibility facts and to the assumed consistency of Δ in order to rule out various basic cases as impossible. All these basic cases involve a derivation of the conclusion that $\Delta \vdash \bot$, contradicting our main assumption. The appeals to CUT are to the rule of CUT for $\bot$. Note that in the top four rows, where $\Delta \vdash \varphi$, one appeals to the *elimination* rule for the dominant operator in φ; but in the bottom four rows, where $\Delta, \varphi \vdash \bot$, one appeals instead to the *introduction* rule:

$\Delta \nvdash \bot$	$\Delta \vdash \psi$ $\Delta \vdash \theta$	$\Delta \vdash \psi$ $\Delta, \theta \vdash \bot$	$\Delta, \psi \vdash \bot$ $\Delta \vdash \theta$	$\Delta, \psi \vdash \bot$ $\Delta, \theta \vdash \bot$
$\Delta \vdash \neg\psi$	By $\neg$-E, $\Delta, \neg\psi \vdash \bot$ By CUT, $\Delta \vdash \bot$	By $\neg$-E, $\Delta, \neg\psi \vdash \bot$ By CUT, $\Delta \vdash \bot$		
$\Delta \vdash \psi \wedge \theta$		By $\wedge$-E, $\Delta, \psi \wedge \theta \vdash \bot$ By CUT, $\Delta \vdash \bot$	By $\wedge$-E, $\Delta, \psi \wedge \theta \vdash \bot$ By CUT, $\Delta \vdash \bot$	By $\wedge$-E, $\Delta, \psi \wedge \theta \vdash \bot$ By CUT, $\Delta \vdash \bot$
$\Delta \vdash \psi \vee \theta$				By $\vee$-E, $\Delta, \psi \vee \theta \vdash \bot$ By CUT, $\Delta \vdash \bot$
$\Delta \vdash \psi \rightarrow \theta$		By $\rightarrow$-E, $\Delta, \psi \rightarrow \theta \vdash \bot$ By CUT, $\Delta \vdash \bot$		
$\Delta, \neg\psi \vdash \bot$			By $\neg$-I, $\Delta \vdash \neg\psi$ By CUT, $\Delta \vdash \bot$	By $\neg$-I, $\Delta \vdash \neg\psi$ By CUT, $\Delta \vdash \bot$
$\Delta, \psi \wedge \theta \vdash \bot$	By $\wedge$-I, $\Delta \vdash \psi \wedge \theta$ By CUT, $\Delta \vdash \bot$			
$\Delta, \psi \vee \theta \vdash \bot$	By $\vee$-I, $\Delta \vdash \psi \vee \theta$ By CUT, $\Delta \vdash \bot$	By $\vee$-I, $\Delta \vdash \psi \vee \theta$ By CUT, $\Delta \vdash \bot$	By $\vee$-I, $\Delta \vdash \psi \vee \theta$ By CUT, $\Delta \vdash \bot$	
$\Delta, \psi \rightarrow \theta \vdash \bot$	By $\rightarrow$-I, $\Delta \vdash \psi \rightarrow \theta$ By CUT, $\Delta \vdash \bot$		By $\rightarrow$-I, $\Delta \vdash \psi \rightarrow \theta$ By CUT, $\Delta \vdash \bot$	By $\rightarrow$-I, $\Delta \vdash \psi \rightarrow \theta$ By CUT, $\Delta \vdash \bot$

With those basic cases out of the way (now indicated in the following picture by crosses in the boxes concerned), the remaining basic cases involve appeal to the Inductive Hypothesis (parts (i) and (ii)) concerning the subsentences ψ and θ, and to the rules of evaluation. Note that one appeals to the *verification* rule for the dominant operator in φ when $\Delta \vdash \varphi$, but to the

falsification rule when $\Delta, \varphi \vdash \bot$:

$\Delta \nvdash \bot$	$\Delta \vdash \psi$ $\Delta \vdash \theta$	$\Delta \vdash \psi$ $\Delta, \theta \vdash \bot$	$\Delta, \psi \vdash \bot$ $\Delta \vdash \theta$	$\Delta, \psi \vdash \bot$ $\Delta, \theta \vdash \bot$
$\Delta \vdash \neg\psi$			By IH(ii) $\lvert\Delta\rvert, \psi \Vdash \bot$ By$\neg\mathcal{V}$, $\lvert\Delta\rvert \Vdash \neg\psi$	By IH(ii) $\lvert\Delta\rvert, \psi \Vdash \bot$ By$\neg\mathcal{V}$, $\lvert\Delta\rvert \Vdash \neg\psi$
$\Delta \vdash \psi \wedge \theta$	By IH(i), $\lvert\Delta\rvert \Vdash \psi$ By IH(i), $\lvert\Delta\rvert \Vdash \theta$ By $\wedge\mathcal{V}$, $\lvert\Delta\rvert \Vdash \psi \wedge \theta$			
$\Delta \vdash \psi \vee \theta$	By IH(i), $\lvert\Delta\rvert \Vdash \psi$ By $\vee\mathcal{V}$, $\lvert\Delta\rvert \Vdash \psi \vee \theta$	By IH(i), $\lvert\Delta\rvert \Vdash \psi$ By $\vee\mathcal{V}$, $\lvert\Delta\rvert \Vdash \psi \vee \theta$	By IH(i), $\lvert\Delta\rvert \Vdash \theta$ By $\vee\mathcal{V}$, $\lvert\Delta\rvert \Vdash \psi \vee \theta$	
$\Delta \vdash \psi \to \theta$	By IH(i), $\lvert\Delta\rvert \Vdash \theta$ By$\to\mathcal{V}$, $\lvert\Delta\rvert \Vdash \psi \to \theta$		By IH(i), $\lvert\Delta\rvert \Vdash \theta$ By$\to\mathcal{V}$, $\lvert\Delta\rvert \Vdash \psi \to \theta$	By IH(ii) $\lvert\Delta\rvert, \psi \Vdash \bot$ By$\to\mathcal{V}$, $\lvert\Delta\rvert \Vdash \psi \to \theta$
$\Delta, \neg\psi \vdash \bot$	By IH(i), $\lvert\Delta\rvert \Vdash \psi$ By $\neg\mathcal{F}$ $\lvert\Delta\rvert, \neg\psi \Vdash \bot$	By IH(i), $\lvert\Delta\rvert \Vdash \psi$ By $\neg\mathcal{F}$ $\lvert\Delta\rvert, \neg\psi \Vdash \bot$		
$\Delta, \psi \wedge \theta \vdash \bot$		By IH(ii) $\lvert\Delta\rvert, \theta \Vdash \bot$ By$\wedge\mathcal{F}$ $\lvert\Delta\rvert, \psi \wedge \theta \Vdash \bot$	By IH(ii) $\lvert\Delta\rvert, \psi \Vdash \bot$ By$\wedge\mathcal{F}$ $\lvert\Delta\rvert, \psi \wedge \theta \Vdash \bot$	By IH(ii) $\lvert\Delta\rvert, \psi \Vdash \bot$ By$\wedge\mathcal{F}$ $\lvert\Delta\rvert, \psi \wedge \theta \Vdash \bot$
$\Delta, \psi \vee \theta \vdash \bot$				By IH(ii) $\lvert\Delta\rvert, \psi \Vdash \bot$ By IH(ii) $\lvert\Delta\rvert, \theta \Vdash \bot$ By $\vee\mathcal{F}$ $\lvert\Delta\rvert, \psi \vee \theta \Vdash \bot$
$\Delta, \psi \to \theta \vdash \bot$		By IH(i) $\lvert\Delta\rvert \Vdash \psi$ By IH(ii) $\lvert\Delta\rvert, \theta \Vdash \bot$ By $\to\mathcal{F}$, $\lvert\Delta\rvert, \psi \to \theta \Vdash \bot$		

When we restore the full workings for the earlier basic cases, the picture is as follows:

$\Delta \nvdash \bot$	$\Delta \vdash \psi$ $\Delta \vdash \theta$	$\Delta \vdash \psi$ $\Delta, \theta \vdash \bot$	$\Delta, \psi \vdash \bot$ $\Delta \vdash \theta$	$\Delta, \psi \vdash \bot$ $\Delta, \theta \vdash \bot$
$\Delta \vdash \neg\psi$	By $\neg$-E, $\Delta, \neg\psi \vdash \bot$ By CUT, $\Delta \vdash \bot$	By $\neg$-E, $\Delta, \neg\psi \vdash \bot$ By CUT, $\Delta \vdash \bot$	By IH(ii) $\lvert\Delta\rvert, \psi \Vdash \bot$ By$\neg\mathcal{V}$, $\lvert\Delta\rvert \Vdash \neg\psi$	By IH(ii) $\lvert\Delta\rvert, \psi \Vdash \bot$ By$\neg\mathcal{V}$, $\lvert\Delta\rvert \Vdash \neg\psi$
$\Delta \vdash \psi \wedge \theta$	By IH(i), $\lvert\Delta\rvert \Vdash \psi$ By IH(i), $\lvert\Delta\rvert \Vdash \theta$ By $\wedge\mathcal{V}$, $\lvert\Delta\rvert \Vdash \psi \wedge \theta$	By $\wedge$-E, $\Delta, \psi \wedge \theta \vdash \bot$ By CUT, $\Delta \vdash \bot$	By $\wedge$-E, $\Delta, \psi \wedge \theta \vdash \bot$ By CUT, $\Delta \vdash \bot$	By $\wedge$-E, $\Delta, \psi \wedge \theta \vdash \bot$ By CUT, $\Delta \vdash \bot$
$\Delta \vdash \psi \vee \theta$	By IH(i), $\lvert\Delta\rvert \Vdash \psi$ By $\vee\mathcal{V}$, $\lvert\Delta\rvert \Vdash \psi \vee \theta$	By IH(i), $\lvert\Delta\rvert \Vdash \psi$ By $\vee\mathcal{V}$, $\lvert\Delta\rvert \Vdash \psi \vee \theta$	By IH(i), $\lvert\Delta\rvert \Vdash \theta$ By $\vee\mathcal{V}$, $\lvert\Delta\rvert \Vdash \psi \vee \theta$	By $\vee$-E, $\Delta, \psi \vee \theta \vdash \bot$ By CUT, $\Delta \vdash \bot$
$\Delta \vdash \psi \to \theta$	By IH(i), $\lvert\Delta\rvert \Vdash \theta$ By$\to\mathcal{V}$, $\lvert\Delta\rvert \Vdash \psi \to \theta$	By$\to$-E, $\Delta, \psi \to \theta \vdash \bot$ By CUT, $\Delta \vdash \bot$	By IH(i), $\lvert\Delta\rvert \Vdash \theta$ By$\to\mathcal{V}$, $\lvert\Delta\rvert \Vdash \psi \to \theta$	By IH(ii) $\lvert\Delta\rvert, \psi \Vdash \bot$ By$\to\mathcal{V}$, $\lvert\Delta\rvert \Vdash \psi \to \theta$
$\Delta, \neg\psi \vdash \bot$	By IH(i), $\lvert\Delta\rvert \Vdash \psi$ By $\neg\mathcal{F}$ $\lvert\Delta\rvert, \neg\psi \Vdash \bot$	By IH(i), $\lvert\Delta\rvert \Vdash \psi$ By $\neg\mathcal{F}$ $\lvert\Delta\rvert, \neg\psi \Vdash \bot$	By$\neg$I, $\Delta \vdash \neg\psi$ By CUT, $\Delta \vdash \bot$	By$\neg$I, $\Delta \vdash \neg\psi$ By CUT, $\Delta \vdash \bot$
$\Delta, \psi \wedge \theta \vdash \bot$	By $\wedge$-I, $\Delta \vdash \psi \wedge \theta$ By CUT, $\Delta \vdash \bot$	By IH(ii) $\lvert\Delta\rvert, \theta \Vdash \bot$ By$\wedge\mathcal{F}$ $\lvert\Delta\rvert, \psi \wedge \theta \Vdash \bot$	By IH(ii) $\lvert\Delta\rvert, \psi \Vdash \bot$ By$\wedge\mathcal{F}$ $\lvert\Delta\rvert, \psi \wedge \theta \Vdash \bot$	By IH(ii) $\lvert\Delta\rvert, \psi \Vdash \bot$ By$\wedge\mathcal{F}$ $\lvert\Delta\rvert, \psi \wedge \theta \Vdash \bot$
$\Delta, \psi \vee \theta \vdash \bot$	By $\vee$-I, $\Delta \vdash \psi \vee \theta$ By CUT, $\Delta \vdash \bot$	By $\vee$-I, $\Delta \vdash \psi \vee \theta$ By CUT, $\Delta \vdash \bot$	By $\vee$-I, $\Delta \vdash \psi \vee \theta$ By CUT, $\Delta \vdash \bot$	By IH(ii) $\lvert\Delta\rvert, \psi \Vdash \bot$ By IH(ii) $\lvert\Delta\rvert, \theta \Vdash \bot$ By $\vee\mathcal{F}$ $\lvert\Delta\rvert, \psi \vee \theta \Vdash \bot$
$\Delta, \psi \to \theta \vdash \bot$	By $\to$-I, $\Delta \vdash \psi \to \theta$ By CUT, $\Delta \vdash \bot$	By IH(i) $\lvert\Delta\rvert \Vdash \psi$ By IH(ii) $\lvert\Delta\rvert, \theta \Vdash \bot$ By$\to\mathcal{F}$, $\lvert\Delta\rvert, \psi \to \theta \Vdash \bot$	By $\to$-I, $\Delta \vdash \psi \to \theta$ By CUT, $\Delta \vdash \bot$	By $\to$-I, $\Delta \vdash \psi \to \theta$ By CUT, $\Delta \vdash \bot$