Strong Completeness of Classical Propositional Logic: Summary of Definitions and Results

by

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CUT FOR ABSURDITY: $\dfrac{\Delta_1 \vdash \varphi \quad \Delta_2, \varphi \vdash \bot}{\Delta_1, \Delta_2 \vdash \bot}$ holds for all systems here.

Definition 1 *We write* Δ, φ *for* $\Delta \cup \{\varphi\}$.

Definition 2 Δ decides φ *just in case either* $\Delta \vdash \varphi$ *or* $\Delta, \varphi \vdash \bot$.

Definition 3 Δ *is* decisive *just in case* Δ *decides every sentence in the language of* Δ.

Definition 4 (Expansions) $\begin{cases} \Delta \oplus \varphi =_{df} \Delta, \varphi & \text{if } \Delta, \varphi \nvdash \bot; \\ \Delta \oplus \varphi =_{df} \Delta, \neg\varphi & \text{if } \Delta, \varphi \vdash \bot \end{cases}$

Observation 1 $\Delta \subseteq (\Delta \oplus \varphi)$.

Lemma 1 $\Delta \oplus \varphi$ *decides* φ.

Lemma 2 *If* $\Delta \nvdash \bot$, *then* $\Delta \oplus \varphi \nvdash \bot$.

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Definition 5 *Let Φ be any countable list of sentences $\varphi_0, \varphi_1, \ldots$. Let Δ be any set of sentences. Then*

$$\begin{array}{lll} \Delta_0^\Phi & =_{df} & \Delta \\ \Delta_{n+1}^\Phi & =_{df} & (\Delta_n^\Phi) \oplus \varphi_n \\ \Delta^\Phi & =_{df} & \bigcup_i \Delta_i^\Phi \end{array}$$

Observation 2 $\Delta = \Delta_0^\Phi \subseteq \Delta_1^\Phi \subseteq \ldots \subseteq \Delta_n^\Phi \subseteq \Delta_{n+1}^\Phi \subseteq \ldots \subseteq \Delta^\Phi$.

Lemma 3 *If $\Delta \nvdash \bot$, then for all n, $\Delta_n^\Phi \nvdash \bot$.*

Lemma 4 *If $\Delta \nvdash \bot$, then $\Delta^\Phi \nvdash \bot$.*

Lemma 5 *Δ^Φ decides every member of the list Φ.*

Lemma 6
(i) For any φ in Φ, if $\Delta^\Phi \vdash \varphi$, then φ is in Δ^Φ.
(ii) For any φ in Φ, if $\Delta^\Phi, \varphi \vdash \bot$, then $\neg\varphi$ is in Δ^Φ.

Definition 6 *Let $\mathbb{A}$ be a list of all atoms in the language of Δ.*

Corollary 1 *Let Δ be consistent. Then $\Delta^\mathbb{A}$ is consistent.*

Corollary 2 *$\Delta^\mathbb{A}$ decides every atom in the language of Δ.*

Definition 7 *A* literal *is an atom or the negation of an atom.*

Definition 8 $|\Delta| =_{\text{df}}$ *the set of literals deducible from Δ.*

Corollary 3 *Every literal deducible from $\Delta^\mathbb{A}$ is in $\Delta^\mathbb{A}$.*

Definition 9 *Let τ be a set of literals. We say that τ* deals with *φ just in case for each atom A in φ, either A or $\neg A$ is in τ (but not both).*

Lemma 7 *Suppose that the set τ of literals deals with φ. Then either $\tau \vdash \varphi$ or $\tau, \varphi \vdash \bot$.*

Theorem 1 *$\Delta^\mathbb{A}$ is decisive.*

Note that $\Vdash\ \subseteq\ \vdash\ \subseteq\ \vdash_I\ \subseteq\ \vdash_C$.

Definition 10 $|\Delta|$ *is the set of literals deducible from* Δ*, i.e.* $\{\lambda \mid \Delta \vdash \lambda\}$*.*

Theorem 2 *Suppose* Δ *is consistent and decisive. Then for every sentence* φ*:*
(i) if $\Delta \vdash \varphi$*, then* $|\Delta| \Vdash \varphi$*; and*
(ii) if $\Delta, \varphi \vdash \bot$*, then* $|\Delta|, \varphi \Vdash \bot$*.*

Theorem 3 (Satisfiability)
If Δ *is consistent, then for every member* φ *of* Δ *we have* $|\Delta^{\mathbb{A}}| \Vdash \varphi$*.*

Definition 11 *We say* Δ (classically) logically implies φ*, or* φ is a (classical) logical consequence of Δ*, and write* $\Delta \models \varphi$*, just in case every coherent set of literals that verifies every member of* Δ *verifies* φ *(that is: for every coherent* τ *if* $\tau \Vdash \Delta$*, then* $\tau \Vdash \varphi$*). We write* $\Delta \models \bot$ *just in case there is no coherent set* τ *of literals such that* $\tau \Vdash \Delta$*.*

Lemma 8 *If* $\Delta \models \varphi$*, then* $\Delta, \neg\varphi \models \bot$*.*

Definition 12 *We write* $\vdash_C$ *for the deducibility relation extending* $\vdash$ *by being closed under the rule of* classical reductio:

$$\frac{\Delta, \neg\varphi \vdash \bot}{\Delta \vdash \varphi}$$

Theorem 4 (Strong Completeness of Classical Propositional Logic)
If $\Delta \models \varphi$*, then* $\Delta \vdash_C \varphi$*.*

Lemma 9 (Double Negation Lemma) $\neg\neg\psi \models \psi$*.*

Definition 13 $\neg\neg\Gamma =_{\mathrm{df}} \{\neg\neg\gamma | \gamma \in \Gamma\}$*.*

Observation 3 *If* $\neg\neg\Gamma \vdash \bot$ *then (since* $\gamma \vdash \neg\neg\gamma$*) we have* $\Gamma \vdash \bot$*. So by contraposition, if* Γ *is consistent, then* $\neg\neg\Gamma$ *is too.*

Theorem 5 (Gödel–Gentzen–Glivenko, modified)
If $\Gamma \nvdash \bot$ *and* $\Gamma \models \psi$ *then* $\neg\neg\Gamma \vdash \neg\neg\psi$*.*