Strong Completeness of Classical Propositional Logic based on Sheffer's Stroke

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1 Introduction

The Sheffer stroke $|$ is a binary connective. $\varphi \,|\, \psi$ means 'It is not the case that both φ and ψ'. So $|$ has the following truth table:

φ	ψ	$\varphi \| \psi$
T	T	F
T	F	T
F	T	T
F	F	T

and it is governed by the following rules of inference:[1]

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[1] See N. Tennant, 'La barre de Sheffer dans la logique des sequents et des syllogismes', *Logique et Analyse* 88, 1979, pp. 503-514. The only difference here is that the introduction rule is relevantized by making discharge of assumptions obligatory.

$$
|\text{-I}\quad \dfrac{\begin{array}{c} \underbrace{\overset{(i)\,\overline{\quad}\,\Box\,\overline{\quad}\,(i)}{\varphi \quad \psi}} \\ \vdots \\ \bot \end{array}}{\varphi \,|\, \psi}\,(i)
\qquad\qquad
|\text{-E}\quad \dfrac{\varphi \quad \varphi \,|\, \psi \quad \psi}{\bot}
\qquad\qquad
\text{CR}\quad \dfrac{\begin{array}{c} \overline{\varphi \,|\, \varphi}^{\,(i)} \\ \vdots \\ \bot \end{array}}{\varphi}\,(i)
$$

With the rule (CR) the discharge of assumptions is permissive, not obligatory. That is, there does not have to be an assumption of the indicated form within the sub-proof; but, if there is, then it is discharged (at all its occurrences) upon applying the rule in question. With the rule (|-I), however, the discharge of assumptions is obligatory (as indicated by the box): there must be an undischarged assumption of at least one of the indicated forms within the sub-proof; and all such assumption-occurrences are discharged upon applying the rule in question.

The rules (|-I) and (|-E) form the system of *intuitionistic relevant logic* for the Sheffer stroke. By adding the rule (CR) of *classical reductio* one obtains full *classical logic*. All the usual connectives can be defined in terms of the Sheffer stroke:

$$
\begin{array}{lll}
\neg\varphi & =_{df} & \varphi \mid \varphi \\
\varphi \wedge \psi & =_{df} & (\varphi \mid \psi) \mid (\varphi \mid \psi) \\
\varphi \vee \psi & =_{df} & (\varphi \mid \varphi) \mid (\psi \mid \psi) \\
\varphi \rightarrow \psi & =_{df} & \varphi \mid (\psi \mid \psi)
\end{array}
$$

It follows that by proving the strong completeness of the system based on the Sheffer stroke alone, we prove the strong completeness of full classical logic.

2 Syntax

2.1 Some system-invariant results

We begin by establishing some results that involve the general notion of deducibility—for which we write $\vdash$—and the notion of negation. We shall write $\overline{\varphi}$ for the negation of φ. For the purposes of this subsection, it is required only that if $\Delta, \varphi \vdash \bot$ then $\Delta \vdash \overline{\varphi}$.

Definition 1 *We write Δ, φ for $\Delta \cup \{\varphi\}$.*

Definition 2 *A* literal *is an atom or the negation of an atom.*

We use τ to denote sets of literals.

Definition 3 *A set of literals is* coherent *just in case it does not contain the negation of any atom that it contains.*

Definition 4 Δ decides φ *just in case either* $\Delta \vdash \varphi$ *or* $\Delta, \varphi \vdash \bot$.

Definition 5 Δ *is* decisive *just in case* Δ *decides every sentence in the language of* Δ.

Definition 6 *The rule of* CUT FOR $\bot$ *is* $\dfrac{\Delta_1 \vdash \varphi \quad \Delta_2, \varphi \vdash \bot}{\Delta_1, \Delta_2 \vdash \bot}$

The rule of CUT for $\bot$ holds *for* our proof system; it need not be a rule *of* that system itself. The CUT rule is a correct metalinguistic inference *about* deducibility in the object language.[2] This relation of deducibility can be defined by reference to quite different rules of inference, which make up the proof system for the object-logic in question.

Definition 7 (Expansions) $\begin{cases} \Delta \oplus \varphi =_{df} \Delta, \varphi & \text{if } \Delta, \varphi \nvdash \bot; \\ \Delta \oplus \varphi =_{df} \Delta, \overline{\varphi} & \text{if } \Delta, \varphi \vdash \bot \end{cases}$

Observation 1 $\Delta \subseteq (\Delta \oplus \varphi)$.

Lemma 1 $\Delta \oplus \varphi$ *contains either* φ *or* $\overline{\varphi}$.

Proof. Obvious.

Lemma 2 *If* $\Delta \nvdash \bot$, *then* $\Delta \oplus \varphi \nvdash \bot$.

Proof. Suppose $\Delta \nvdash \bot$, and, for the main *reductio*, that $\Delta \oplus \varphi \vdash \bot$. Suppose, for subsidiary *reductio*, that $\Delta, \varphi \vdash \bot$. Then $\Delta \vdash \overline{\varphi}$. Also, by definition $\Delta \oplus \varphi = \Delta, \overline{\varphi}$. Hence $\Delta, \overline{\varphi} \vdash \bot$. So, by CUT for $\bot$, we have $\Delta \vdash \bot$. Contradiction. So $\Delta, \varphi \nvdash \bot$ (discharging the subsidiary *reductio* assumption). It now follows by definition that $\Delta \oplus \varphi = \Delta, \varphi$. Hence $\Delta, \varphi \vdash \bot$. Contradiction. Hence $\Delta \oplus \varphi \nvdash \bot$, discharging the main *reductio* assumption.

[2] CUT FOR $\bot$ is a corollary to a metatheorem about the system *IR* of intuitionistic relevant logic (whatever the choice of connectives). The metatheorem in question states

If $\Delta_1 \vdash \varphi$ and $\Delta_2, \varphi \vdash \psi$, then either $\Delta_1, \Delta_2 \vdash \psi$ or $\Delta_1, \Delta_2 \vdash \bot$.

The corollary in question is obtained by taking $\bot$ for ψ. For then one obtains

If $\Delta_1 \vdash \varphi$ and $\Delta_2, \varphi \vdash \bot$, then $\Delta_1, \Delta_2 \vdash \bot$.

Definition 8 *Let Φ be any countable list of sentences $\varphi_0, \varphi_1, \ldots$. Let Δ be any set of sentences. Then*

$$\begin{array}{lll} \Delta_0^\Phi & =_{df} & \Delta \\ \Delta_{n+1}^\Phi & =_{df} & (\Delta_n^\Phi) \oplus \varphi_n \\ \Delta^\Phi & =_{df} & \bigcup_i \Delta_i^\Phi \end{array}$$

Observation 2 $\Delta = \Delta_0^\Phi \subseteq \Delta_1^\Phi \subseteq \ldots \subseteq \Delta_n^\Phi \subseteq \Delta_{n+1}^\Phi \subseteq \ldots \subseteq \Delta^\Phi$.

Lemma 3 *If $\Delta \nvdash \bot$, then for all n, $\Delta_n^\Phi \nvdash \bot$.*

Proof. Immediate by mathematical induction from the hypothesis of the Lemma (basis step), and Lemma 2 (inductive step).

Lemma 4 *If $\Delta \nvdash \bot$, then $\Delta^\Phi \nvdash \bot$.*

Proof. Suppose $\Delta \nvdash \bot$. Suppose for *reductio* that $\Delta^\Phi \vdash \bot$. Proof is finite. Hence for some m we have $\Delta_m^\Phi \vdash \bot$, contradicting Lemma 3. Hence $\Delta^\Phi \nvdash \bot$.

Lemma 5 *For every member φ of the list Φ, Δ^Φ contains either φ or $\overline{\varphi}$.*

Proof. Consider any member φ_n of the list. By Lemma 1, Δ_{n+1}^Φ contains either φ_n or $\overline{\varphi_n}$. Hence so does Δ^Φ.

Lemma 6 *Suppose $\Delta \nvdash \bot$.*
(i) For any φ in Φ, if $\Delta^\Phi \vdash \varphi$, then φ is in Δ^Φ.
(ii) For any φ in Φ, if $\Delta^\Phi, \varphi \vdash \bot$, then $\overline{\varphi}$ is in Δ^Φ.

Proof. Let φ occur on the list Φ as φ_k.
(i) Suppose $\Delta^\Phi \vdash \varphi$. By consistency of Δ^Φ (Lemma 4) and CUT for $\bot$, we have $\Delta_k^\Phi, \varphi \nvdash \bot$. By definition of $\oplus$, $\Delta_{k+1}^\Phi = \Delta_k^\Phi \oplus \varphi = \Delta_k^\Phi, \varphi$. Hence φ is in Δ^Φ.
(ii) Suppose $\Delta^\Phi, \varphi \vdash \bot$. By consistency of Δ^Φ (Lemma 4) and CUT for $\bot$, φ is not in Δ_{k+1}^Φ. So by definition of $\oplus$, we have that $\Delta_k^\Phi, \varphi \vdash \bot$. But then by definition of $\oplus$ again, we have $\Delta_{k+1}^\Phi = \Delta_k^\Phi \oplus \varphi = \Delta_k^\Phi, \overline{\varphi}$. Hence $\overline{\varphi}$ is in Δ^Φ.

Definition 9 *Let $\mathbb{A}$ be a list of all atoms in the language of Δ.*

Corollary 1 *Let Δ be consistent. Then $\Delta^\mathbb{A}$ is consistent.*

Proof. Special case of Lemma 4.

Corollary 2 *For every atom A, $\Delta^\mathbb{A}$ contains either A or $\overline{A}$.*

Proof. Special case of Lemma 5.

Definition 10 $|\Delta| =_{\text{df}}$ *the set of literals deducible from* Δ.

Definition 11 *Let* τ *be a set of literals. We say that* τ deals with φ *just in case for each atom* A *in* φ, *either* A *or* $\overline{A}$ *is in* τ *(but not both).*

2.2 Some system-specific results

From now on $\vdash$ stands for *intuitionistic relevant* deducibility, i.e. deducibility in accordance with the introduction and elimination rules (with discharge conventions ensuring relevance of premises to conclusions). In this study, the system consists of the rules ($|$-I) and ($|$-E) for the Sheffer stroke, as explained above. In the Sheffer system, the *negation* of φ is $(\varphi\,|\,\varphi)$. All results of the previous subsection apply.

Lemma 7 *Suppose that the set* τ *of literals deals with* φ. *Then either* $\tau \vdash \varphi$ *or* $\tau, \varphi \vdash \bot$.

Proof. By induction on φ.
Basis. Suppose φ is an atom A. By main supposition, either A or $A\,|\,A$ is in τ. If A is in τ, then $\tau \vdash A$. If $A\,|\,A$ is in τ, then by ($|$-E) we have $\tau, A \vdash \bot$. So either way, the result holds for atomic φ.
Inductive Hypothesis. Suppose the result holds for ψ and θ.
Inductive Step. We show that the result holds for $\psi \mid \theta$. We note that it follows from our main supposition that for every atom A involved in ψ or in θ, exactly one of A or $A\,|\,A$ is in the set τ of literals. By IH there are four cases to consider:
(i) $\tau \vdash \psi$ and $\tau \vdash \theta$;
(ii) $\tau \vdash \psi$ and $\tau, \theta \vdash \bot$;
(iii) $\tau, \psi \vdash \bot$ and $\tau \vdash \theta$;
(iv) $\tau, \psi \vdash \bot$ and $\tau, \theta \vdash \bot$.
In case (i), by ($|$-E) it follows that $\tau, \psi \mid \theta \vdash \bot$. In cases (ii), (iii) and (iv) the conjuncts involving $\bot$ ensure, by ($|$-I), that $\tau \vdash \psi\,|\,\theta$. *QED*

Theorem 1 *If* Δ *is consistent, then* $\Delta^{\mathbb{A}}$ *is decisive.*

Proof. Let Δ be consistent. By Corollary 2, $\Delta^{\mathbb{A}}$ includes a set of literals (τ, say) that deals with every sentence. By Lemma 7, τ decides all sentences. So $\Delta^{\mathbb{A}}$ is decisive.

3 Inferential Semantics

For every truth-value assignment $\boldsymbol{\tau}$ (note the boldface) there is an obvious coherent set τ of literals (note the lightface) that codifies it:

$$A \in \tau \Leftrightarrow \boldsymbol{\tau}(A) = T;$$

$$(A\,|\,A) \in \tau \Leftrightarrow \boldsymbol{\tau}(A) = F.$$

Consider the following rules for verifying and for falsifying sentences *modulo* a set of literals. We call them $\mathcal{V}$- and $\mathcal{F}$-rules. No proof-work is allowed above the major premise $\varphi_1\,|\,\varphi_2$ of the rule $(|\ \mathcal{F})$. As indicated by the box, the discharge in the rule $(|\ \mathcal{V})$ is obligatory: that is, one must have used an assumption of the indicated form within the sub-proof.

$$(\alpha\mathcal{F}) \qquad \frac{A\,|\,A \qquad A}{\bot}$$

$$(|\ \mathcal{V}) \qquad \frac{\begin{array}{c} \tau\ ,\ \overset{\Box\text{——}(i)}{\varphi_k} \\ \vdots \\ \bot \end{array}}{\varphi_1\,|\,\varphi_2}\,(i) \qquad (k = 1, 2) \qquad\qquad (|\ \mathcal{F}) \qquad \frac{\varphi_1\,|\,\varphi_2 \quad \begin{array}{c} \tau_1 \\ \vdots \\ \varphi_1 \end{array} \quad \begin{array}{c} \tau_2 \\ \vdots \\ \varphi_2 \end{array}}{\bot}$$

Definition 12 *$\Vdash$ is the 'deducibility' relation generated by the $\mathcal{V}$- and $\mathcal{F}$-rules. But we can read $\tau \Vdash \varphi$ as 'τ makes φ true', and read $\tau, \varphi \Vdash \bot$ as 'τ makes φ false'.*

Lemma 8 (Incoherence Lemma) *Suppose $\tau \Vdash \varphi$ and $\tau, \varphi \Vdash \bot$. Then τ is incoherent; i.e., for some atom A, both A and $A\,|\,A$ are in τ.*

Proof. By induction on φ.
Basis. Suppose $\tau \Vdash A$ and $\tau, A \Vdash \bot$. The only way for $\tau \Vdash A$ to hold is for A to be in τ. The only way for $\tau, A \Vdash \bot$ to hold is by dint of $(\alpha\mathcal{F})$, with $A\,|\,A \in \tau$. So τ is incoherent.
Inductive Hypothesis. Suppose the result holds for ψ and θ.
Inductive Step. We show that the result holds for $\psi\,|\,\theta$. So suppose that $\tau \Vdash \psi\,|\,\theta$ and $\tau, \psi\,|\,\theta \Vdash \bot$. We now show that τ is incoherent.

By inspection of the rule $(|\ \mathcal{V})$, the only way for $\tau \Vdash \psi\,|\,\theta$ to hold is that either $\tau, \psi \Vdash \bot$ or $\tau, \theta \Vdash \bot$.

So suppose that $\tau, \psi \Vdash \bot$. By inspection of the rule ($| \mathcal{F}$), the only way for $\tau, \psi | \theta \Vdash \bot$ to hold is that both $\tau \Vdash \psi$ hold and $\tau \Vdash \theta$ hold. So we have both $\tau \Vdash \psi$ and $\tau, \psi \Vdash \bot$. Hence by IH τ is incoherent.

The argument in the case where $\tau, \theta \Vdash \bot$ is similar. *QED*

Comment. Lemma 8 is an inferential-semantic special case of the rule of CUT for $\bot$, if we read $\Vdash$ as $\vdash$ (in the system *IR*). Indeed, the rule of CUT for $\bot$ can be thought of as *arrived at* by generalizing Lemma 8 so as to have arbitrary sets Δ of sentences in place of sets τ of literals (and reading $\Vdash$ as $\vdash$).

Corollary 3 (Monoletheism) *Let τ be a coherent set of literals. Then for all φ not both $\tau \Vdash \varphi$ and $\tau, \varphi \Vdash \bot$.*

Proof. Immediate by contraposition on Lemma 8.

Corollary 3 is the inferential-semantic way of saying that no truth-value assignment makes any sentence both true and false.

Definition 13 *We say* Δ (classically) logically implies φ, *or* φ is a (classical) logical consequence of Δ, *and write $\Delta \models \varphi$, just in case every coherent set of literals that verifies every member of Δ verifies φ (that is: for every coherent τ if $\tau \Vdash \Delta$, then $\tau \Vdash \varphi$). We write $\Delta \models \bot$ just in case there is no coherent set τ of literals such that $\tau \Vdash \Delta$.*

Lemma 9 *If $\Delta \models \varphi$, then $\Delta, \varphi | \varphi \models \bot$.*

Proof. Suppose $\Delta \models \varphi$. Then:

for every coherent τ if $\tau \Vdash \Delta$, then $\tau \Vdash \varphi \ldots$ (1).

Now suppose for *reductio* that there is some coherent τ' such that $\tau' \Vdash \Delta$ and $\tau' \Vdash \varphi | \varphi$. The only way to secure the latter is to have

$$\tau', \varphi \Vdash \bot,$$

and use ($| \mathcal{V}$). By (1) we have also

$$\tau' \Vdash \varphi.$$

But by Corollary 3 (Monoletheism) the last two displayed conclusions contradict each other. So there is no coherent τ' such that $\tau' \Vdash \Delta$ and $\tau' \Vdash \varphi | \varphi$. *QED*

Note that the converse of Lemma 9 also holds, but will not be needed.

4 Combining syntax and (inferential) semantics

Theorem 2 *Suppose Δ is consistent and decisive. Then for every sentence φ:*
(i) if $\Delta \vdash \varphi$, then $|\Delta| \Vdash \varphi$; and
(ii) if $\Delta, \varphi \vdash \bot$, then $|\Delta|, \varphi \Vdash \bot$.

Proof. By induction on φ.
Basis. Suppose φ is atomic. For (i), suppose $\Delta \vdash \varphi$. Then $\varphi \in |\Delta|$, whence obviously $|\Delta| \Vdash \varphi$. For (ii), suppose $\Delta, \varphi \vdash \bot$. By (|)-I we have $\Delta \vdash \varphi | \varphi$. But φ is atomic. So $(\varphi | \varphi) \in |\Delta|$. By rule (α) we have $|\Delta|, \varphi \Vdash \bot$.

Inductive Hypothesis (IH). Suppose (i) and (ii) hold for ψ and for θ.

Inductive Step. For (i), suppose $\Delta \vdash \psi | \theta$. We seek to show that $|\Delta| \Vdash \psi | \theta$. So suppose for *reductio* that $|\Delta| \not\Vdash \psi | \theta$. By the rule $(| \mathcal{V})$, it follows that $|\Delta|, \psi \not\Vdash \bot$ and $|\Delta|, \theta \not\Vdash \bot$. By IH(ii), we have $\Delta, \psi \nvdash \bot$ and $\Delta, \theta \nvdash \bot$. Since Δ is decisive, it follows that $\Delta \vdash \psi$ and $\Delta \vdash \theta$. But by (|-E) we have $\psi \mid \theta, \psi, \theta \vdash \bot$. Hence by three appeals to CUT for $\bot$, we have $\Delta \vdash \bot$, contradicting the consistency of Δ. Hence $|\Delta| \Vdash \psi | \theta$ after all.

For (ii), suppose $\Delta, \psi | \theta \vdash \bot$. We seek to show that $|\Delta|, \psi | \theta \Vdash \bot$. Suppose for *reductio* that $\Delta, \psi \vdash \bot$. Then by (|-I) it would follow that $\Delta \vdash \psi | \theta$. Hence by CUT for $\bot$ we would have $\Delta \vdash \bot$, contradicting the consistency of Δ. So $\Delta, \psi \nvdash \bot$. But Δ is decisive. Thus $\Delta \vdash \psi$. By IH(i) we have $|\Delta| \Vdash \psi$. A similar argument establishes that $|\Delta| \Vdash \theta$. Thus by $(| \mathcal{F})$ we have $|\Delta|, \psi | \theta \Vdash \bot$. *QED*

Theorem 2 tells us that every sentence deducible from a consistent and decisive set Δ is verified by the literals deducible from Δ. It is not required that Δ itself *contain* any literals.

Theorem 3 *If Δ is consistent, then for every member φ of Δ we have $|\Delta^{\mathbb{A}}| \Vdash \varphi$.*

Proof. Suppose Δ is consistent. By Corollary 1 and Theorem 1, $\Delta^{\mathbb{A}}$ is consistent and decisive. Hence by Theorem 2(i), we have that for every sentence θ, if $\Delta^{\mathbb{A}} \vdash \theta$, then $|\Delta^{\mathbb{A}}| \Vdash \theta$. Let φ be any member of Δ. As observed earlier, $\Delta \subseteq \Delta^{\mathbb{A}}$. So, trivially, $\Delta^{\mathbb{A}} \vdash \varphi$. Hence $|\Delta^{\mathbb{A}}| \Vdash \varphi$. *QED*

Theorem 4 (Strong Completeness of Classical Propositional Logic) *If $\Delta \models \varphi$, then $\Delta \vdash_C \varphi$.*

Proof. Suppose $\Delta \models \varphi$. By Lemma 9, no coherent set of literals verifies all of $\Delta, \varphi | \varphi$. By Theorem 3 we have $\Delta, \varphi | \varphi \vdash \bot$. Hence by the rule (CR) we have $\Delta \vdash_C \varphi$. *QED*