The Non-Compossibility Theorem

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May 2006

Theorem 1 *For any language L and for any set Δ of L-sentences, it is impossible that the following hold:*

1. *S is a sound proof-system for the logical operators in L*
2. *M is an infinite L-model*
3. *every element of M is the denotation of some L-term*
4. *every model of Δ is isomorphic to M*
5. *for any consistent set Γ of non-identities in any name-extension of L, if $\Delta \cup \Gamma$ has no model, then there is an S-proof of $\perp$ from $\Delta \cup \Gamma$.*

Proof. Let L be a language, and let Δ be a set of L-sentences. Suppose (1)-(5) hold. We shall derive a contradiction.

Extend L by adding a new name a. Consider the following consistent set of non-identities in the extended language:

$$\{\neg a = t \mid t \text{ is a term in } L\}.$$

Call this set Γ. Suppose $\Delta \cup \Gamma$ has a model. Call this model N. N is a model of Δ. Hence by (4), N is isomorphic to M. That is, there is a 1-1, structure-preserving mapping from N onto M. Call the mapping f. Let the denotation of a in N be α. By (3), the individual $f(\alpha)$ is denoted in M by some L-term t. Hence by the isomorphism between N and M, the individual α is denoted in N by that same term t. So in N the identity $a = t$ is true. But N is a model of Γ. Hence N makes $\neg a = t$ true. Contradiction. It therefore follows that $\Delta \cup \Gamma$ has no model.

Hence by (5), there is an S-proof of $\perp$ from $\Delta \cup \Gamma$. Let Π be the proof of $\perp$ in question. Π has the form

$$\begin{array}{c}\Theta, \Xi \\ \Pi \\ \bot\end{array}$$

where the undischarged assumptions of Π form a finite set $\Theta \cup \Xi$, with $\Theta \subseteq \Delta$ and $\Xi \subseteq \Gamma$. Since Ξ is a finite set of non-identities of the form specified, let those non-identities be $\neg a = t_1, \ldots, \neg a = t_k$, say.

By (2), there is an element α in M not denoted by any of the terms $t_1, \ldots, t_k$.

Extend the model M of Δ by making the name a denote α. Call the extended model M^*.

Since $\Theta \subseteq \Delta$, M is a model of Θ. Moreover, M^* interprets L exactly the way M does. So M^* is a model of Θ.

By choice of α, all the non-identities $\neg a = t_1, \ldots, \neg a = t_k$ are true in M^*; that is, M^* is a model of Ξ.

It follows that M^* is a model of $\Theta \cup \Xi$. That is, M^* makes the premises of the proof Π true. By (1), the proof Π is sound. Hence the conclusion of Π is true in M^*. But the conclusion of Π is $\bot$, which is false in every model.

Contradiction.