Cut for Core Logic

by

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Abstract

The motivation for *Core Logic* is explained. Its system of proof is set out. It is then shown that, although the system has no Cut rule, its relation of deducibility obeys Cut *with epistemic gain*.

1 The debate over logical reform

There is much dispute over which logic is the right logic—indeed, over whether there could even be such a thing as *the* right logic, rather than a spectrum of logics variously suited for different applications in different areas. *Absolutists* about logic regard the use of the definite article as justified; *pluralists* have their principled doubts. For those who engage in the absolutist debate, those whom we can call the *quietists* are willing to accept the full canon $\boldsymbol{C}$ of classical logic. Their opponents, whom we can call the *reformists*—intuitionists and relevantists prominent among them—argue that certain rules of classical logic lack validity, and have no right to be in the canon.

Intuitionists, on the one hand, originally drew inspiration for their critique of classical logic from the requirements of constructivity in mathematical proof. According to the intuitionist's construal of existence, a mathematical existence claim of the form 'there is a natural number n such that

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$F(n)$’ requires its asserter to be able to provide a justifying instance—a constructively determinable number t for which one can prove (intuitionistically!) that $F(t)$:

$$\frac{F(t)}{\exists x F(x)}$$

This means that one may *not* use the ‘back-door’, or indirect, reasoning that would be available to a classical mathematician, whereby in order to derive the conclusion that there is a natural number n such that $F(n)$, it would be sufficient simply to assume that *no* natural number has the property F, and then (classically!) derive an absurdity from that assumption:

$$\dfrac{\begin{array}{c}\overline{\neg\exists x F(x)}^{\,(i)}\\ \vdots\\ \bot\end{array}}{\exists x F(x)}\,(i)$$

Thus the intuitionists ended up rejecting the rule of *Classical Reductio ad Absurdum* (*CR*):

$$\dfrac{\begin{array}{c}\overline{\neg\varphi}^{\,(i)}\\ \vdots\\ \bot\end{array}}{\varphi}\,(i)$$

and all rules equivalent to it, *modulo* the set of rules that the intuitionist could eventually motivate or justify in a more direct fashion. Among these intuitionistic equivalents of (*CR*) is the *Law of Excluded Middle* (*LEM*):

$$\frac{}{\varphi \vee \neg\varphi}$$

through whose rejection intuitionistic logic is perhaps better known.[1]

But even the intuitionists retained the rule *Ex Falso Quodlibet* (*EFQ*),

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[1] In [12], the present author argued for principles governing one’s choice of logic that would rule out the Nelson systems of so-called ‘constructible falsity’ as the right way to accommodate the canons of constructive proof as these are understood by mathematicians. For these systems do not consist of ’separable’ rules for the individual logical operators. There are no separately stateable rules governing negation in the Nelson systems. The rules all deal with co-occurrences of negation with each of the other operators. See [4] and [1].

also known as the *Absurdity Rule*:

$$\frac{\bot}{\varphi}$$

which allows one to infer *any* conclusion one wishes as soon as one has derived an absurdity. This residual rule within intuitionistic logic ***I*** is anathema to relevantists, since it affords an easy proof of the infamous *Lewis's First Paradox*: $A, \neg A \vdash B$. The proof is

$$\frac{\dfrac{A \quad \neg A}{\bot}}{B}$$

Relevantists refuse to accept Lewis's First Paradox, on the grounds that there need not be any connection in meaning between the sentence A in its premises and its conclusion B. Relevantists regard such a 'lack of relevance' between the premises and conclusions of certain classically approved rules of inference as compromising their claim to genuine validity. Many relevantists are still otherwise *classical* in their orientation, in endorsing (as relevantly valid) such inferences as *Double Negation Elimination* (*DNE*), another intuitionistic equivalent of *CR* (and of *LEM*):

$$\frac{\neg\neg\varphi}{\varphi}$$

The picture that emerges is this:[2]

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[2] Note that core logic is a subsystem of intuitionistic logic ***I***, which is a subsystem of classical logic ***C***. Core logic results from undertaking both intuitionist *and* relevantist reform of ***C***. If one undertakes only relevantist reform of ***C***, then (as argued in [12]), the resulting system is ***CR*** (classical relevant logic).

<table>
<tr><td colspan="5"><i>Is there one correct logic?</i></td></tr>
<tr><td colspan="4">Absolutist: <i>Yes</i></td><td rowspan="8">Pluralist: <i>No</i></td></tr>
<tr><td colspan="4"><i>Is it classical?</i></td></tr>
<tr><td rowspan="6">Quietist: <i>Yes</i>
<i>C</i></td><td colspan="3">Reformist: <i>No</i></td></tr>
<tr><td colspan="3"><i>Is it constructive?</i></td></tr>
<tr><td colspan="2">Intuitionist: <i>Yes</i></td><td><i>No</i></td></tr>
<tr><td colspan="2"><i>Is it relevant?</i></td><td rowspan="3"></td></tr>
<tr><td>Relevantist: <i>Yes</i></td><td><i>No</i></td></tr>
<tr><td><i>Core logic</i></td><td><i>I</i></td></tr>
</table>

2 Core logic

Both reformist camps—intuitionist and relevantist—have variously produced philosophical, methodological, meaning-theoretic and intuitive considerations in support of their respective recommendations for restricting classical logic. Their respective complaints about aspects of classical logic have, however, tended to be orthogonal to one another. The two main lines of logical reform—intuitionistic and relevantist—have been concerned with different shortcomings of classical logic. Intuitionists still commit fallacies of relevance, and relevantists still endorse various strictly classical (non-intuitionistic) modes of inference. The present author, however, endorses both kinds of reform, albeit with slightly different results—especially in the matter of relevance—than have been proposed by other authors. The virtue claimed for core logic is that it combines both kinds of reform. A natural (if unwieldy) label for a system of logic resulting from carrying out both intuitionistic and relevantist reforms would be ‘intuitionistic relevant logic’ (*IR*); and that indeed was the name and label proposed in [8] and [9]. A much better name for the system in question, however, would be ‘core logic’; and that is name we shall use here.

All participants in the debate over logical reform have an eye to the methodological requirements of mathematics and natural science. Two central concerns have been:

> *Does one's logic afford all the mathematical theorems that are needed for application in science?*

and

> *Does one's logic enable one to carry out the most rigorous possible tests of a scientific theory?*

It is not our purpose here to argue for affirmative answers to these questions on behalf of core logic. The arguments have been made elsewhere. [12] established the adequacy of core logic for natural science, by adapting the proof in [7] of the adequacy of minimal logic.[3] [10] exploited the naturalness of core logic for efficient proof-search in computational logic. [11] showed that core logic is adequate for intuitionistic mathematics. [12] gave a meaning-theoretic argument for the claim that core logic is the correct logic. Finally, [13] argues for an important revision-theoretic thesis:

> *Core logic is the minimal inviolable core of logic without any part of which one would not be able to establish the rationality of belief-revision.*

Thus the sequence of different kinds of justification for choice of logic terminates in one that underscores the appropriateness of the new name for the system.

Our task, in this study, is to set out the system of core logic, and establish a surprising cut-elimination result for it.

We should make clear at the outset that the current approach, while rooted firmly in the proof-theoretic tradition deriving from the works of Gentzen [2] and Prawitz [5], nevertheless involves a significant departure from them. The reader will see that the rules of inference below are cast in a form that in effect marries the sequent approach with that of natural deduction. Nodes within a proof-tree are labeled with sentences, as is the case with natural deduction. (In sequent proofs, nodes are labeled less economically with sequents.) But the proof-tree itself has the macrostructural economy of a sequent proof. (Natural deductions do not fare well on

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[3] Note that we do not mean to claim that *only* core logic is adequate for natural science. Other systems—including full classical logic!—are adequate too. As a referee has pointed out, David Miller and Yaroslav Shramko argue that *dual intuitionistic logic* is the logic of Popperian science. See [3] and [6]. We can leave to these authors the task of arguing that the logic they favor is the *sole* logic that is adequate for natural science. As far as we are concerned, we wish only to defend our own proposed logical reforms against the anticipated objection that one might lose some of the logical power that is needed in order to test scientific theories. Provably, one does not.

this score, since they often involve repetitions of whole chunks of proof above multiple occurrences of the same sentence.) The resulting 'hybrid' system of proof—combining the advantages of a sequent system with those of natural deduction—was described in detail in [10].

Another important departure from the former contrast between sequent systems and systems of natural deduction is that in the system presented here, proofs are always in normal form. One is not allowed to use 'cuts' to join together a proof of the conclusion A with another proof in which A occurs as a premise. In the Gentzen and Prawitz systems, one *can* do this. The cut-elimination theorem for a Gentzen system then tells one that any sequent proof containing such cuts can be transformed into a proof (of the same overall result) that contains no cuts. And the normalization theorem of Prawitz gives one an analogous assurance in the case where the proofs are natural deductions.

In the system of core proof presented here, the operation of 'making a cut' does not produce a new core proof at all. But what we show (by means of Theorem1) is that one can manipulate two core proofs that otherwise *could* have been 'stuck together' by means of a prohibited cut, so as to obtain the net effect of the core proof that would then have resulted from eliminating that cut. Indeed, the 'net effect' is a pleasing one: in general, one obtains a core proof of *some (possibly proper) subsequent* of the overall result that the process of cut-elimination on traditional proofs would have vouchsafed.

In the statements of rules that follow, the *boxes* next to discharge strokes indicate that vacuous discharge is not allowed. There must be an assumption of the indicated form available for discharge. (With ($\wedge$-E) and ($\forall$-E) we require only that at least one of the indicated assumptions should have been used, and be available for discharge.) The *diamond* next to the discharge stroke in the second half of ($\rightarrow$-I) indicates that it is not required that the assumption in question should have been used and be available for discharge. But if it is available, then it is discharged.

Graphic Rules for Core Logic

(¬-I)

$$\cfrac{\begin{matrix}\square\text{—}(i)\\ \varphi \\ \vdots \\ \bot\end{matrix}}{\neg\varphi}\,(i)$$

(¬-E)

$$\cfrac{\neg\varphi \qquad \begin{matrix}\vdots\\ \varphi\end{matrix}}{\bot}$$

(∧-I)

$$\cfrac{\begin{matrix}\vdots\\ \varphi\end{matrix} \qquad \begin{matrix}\vdots\\ \psi\end{matrix}}{\varphi\wedge\psi}$$

(∧-E)

$$\cfrac{\varphi\wedge\psi \qquad\qquad \begin{matrix}(i)\text{—}\square\text{—}(i)\\ \underbrace{\varphi\,,\psi}\\ \vdots \\ \theta\end{matrix}}{\theta}\,(i)$$

(∨-I)

$$\cfrac{\begin{matrix}\vdots\\ \varphi\end{matrix}}{\varphi\vee\psi} \qquad \cfrac{\begin{matrix}\vdots\\ \psi\end{matrix}}{\varphi\vee\psi}$$

(∨-E)

$$\cfrac{\varphi\vee\psi \quad \begin{matrix}\square\text{—}(i)\\ \varphi\\ \vdots\\ \theta\end{matrix} \quad \begin{matrix}\square\text{—}(i)\\ \psi\\ \vdots\\ \theta\end{matrix}}{\theta}\,(i) \qquad \cfrac{\varphi\vee\psi \quad \begin{matrix}\square\text{—}(i)\\ \varphi\\ \vdots\\ \bot\end{matrix} \quad \begin{matrix}\square\text{—}(i)\\ \psi\\ \vdots\\ \theta\end{matrix}}{\theta}\,(i) \qquad \cfrac{\varphi\vee\psi \quad \begin{matrix}\square\text{—}(i)\\ \varphi\\ \vdots\\ \theta\end{matrix} \quad \begin{matrix}\square\text{—}(i)\\ \psi\\ \vdots\\ \bot\end{matrix}}{\theta}\,(i)$$

(→-I)

$$\dfrac{\begin{array}{c}\square\rule{1.5em}{0.4pt}(i)\\ \varphi\\ \vdots\\ \bot\end{array}}{\varphi\to\psi}\,(i) \qquad \dfrac{\begin{array}{c}\diamond\rule{1.5em}{0.4pt}(i)\\ \varphi\\ \vdots\\ \psi\end{array}}{\varphi\to\psi}\,(i)$$

(→-E)

$$\dfrac{\varphi\to\psi \qquad \begin{array}{c}\vdots\\ \varphi\end{array} \qquad \begin{array}{c}\square\rule{1.5em}{0.4pt}(i)\\ \psi\\ \vdots\\ \theta\end{array}}{\theta}\,(i)$$

(∃-I)

$$\dfrac{\begin{array}{c}\vdots\\ \varphi^x_t\end{array}}{\exists x\varphi}$$

(∃-E)

$$\dfrac{\exists x\varphi^{\textcircled{\cancel{a}}} \qquad \begin{array}{c}\square\rule{1.5em}{0.4pt}(i)\\ \underbrace{\textcircled{\cancel{a}}\ldots\varphi^x_a\ldots\textcircled{\cancel{a}}}\\ \vdots\\ \psi^{\textcircled{\cancel{a}}}\end{array}}{\psi}\,(i)$$

(∀-I)

$$\dfrac{\begin{array}{c}\textcircled{\cancel{a}}\\ \vdots\\ \varphi\end{array}}{\forall x\varphi^a_x}$$

(∀-E)

$$\dfrac{\forall x\varphi \qquad \begin{array}{c}(i)\rule{1.5em}{0.4pt}\cdots\square\cdots\rule{1.5em}{0.4pt}(i)\\ \underbrace{\varphi^x_{t_1}\ ,\ \ldots\ ,\ \varphi^x_{t_n}}\\ \vdots\\ \theta\end{array}}{\theta}\,(i)$$

(=-I) $$\frac{}{t=t}$$

(=-E) $$\frac{t=u \qquad \begin{array}{c}\vdots\\ \varphi\end{array}}{\psi}\text{ , where } \varphi^t_u=\psi^t_u \text{ and } \varphi\neq\psi.$$

The rules for identity are stated for the formal record, but will henceforth be omitted from consideration.

In core proofs, every major premise for an elimination (MPE) stands proud. So *all core proofs are in normal form.* This is because no MPE stands as the conclusion of an application of the corresponding Introduction rule. But by having MPEs stand proud, we are *also* preventing MPEs from standing as the conclusion of an application of any *Elimination* rule!

This raises the question: how can one ensure transitivity of core proof? Suppose one has core (hence, normal) proofs

$$\begin{array}{c}\Delta\\ \Pi\\ A\end{array} \quad \text{and} \quad \begin{array}{c}A,\Gamma\\ \Sigma\\ \theta\end{array}\ ,$$

where, by virtue of A's being displayed separately, it is to be assumed that $A \notin \Gamma$. In systems of proof allowing accumulation of proof-trees, one would be able to form a proof by grafting a copy of Π onto every undischarged assumption-occurrence of A within Σ:

$$\begin{array}{l}\ \ \Delta\\ \ \ \Pi\\ (A)\ ,\Gamma\\ \quad\ \Sigma\\ \quad\ \theta\end{array}$$

and one would have the assurance that the resulting construction would count as a proof of the overall conclusion θ from the set $\Delta\cup\Gamma$ of combined assumptions. Finite repetitions of the accumulation-operation would also of course be countenanced:

$$\begin{array}{ccc} \Delta_1 & & \Delta_n \\ \Pi_1 & ,\ldots, & \Pi_n \\ A_1 & & A_n \end{array} \quad \text{and} \quad \begin{array}{c} A_1,\ldots,A_n,\Gamma \\ \Sigma \\ \theta \end{array} \quad \text{together yield} \quad \begin{array}{c} \begin{array}{ccc} \Delta_1 & & \Delta_n \\ \Pi_1 & & \Pi_n \\ (A_1) & ,\ldots, & (A_n)\,,\Gamma \end{array} \\ \Sigma \\ \theta \end{array} \quad ,$$

where $A_1, \ldots, A_n \notin \Gamma$. This is what is commonly understood as constituting the *transitivity of proof* (within a system allowing the formation of *ab*normal proofs).

Why the stress on '*ab*normal'? Answer: in the core (hence: normal) proof Σ, the 'cut-sentence' A might stand at one of its undischarged assumption-occurrences as the major premise of an elimination. In a system of *normal* proof, the 'proof-accumulation' given above of Π on top of Σ would therefore not always count as a proof (of θ from $\Delta \cup \Gamma$). *A fortiori*, the 'proof-accumulation' just indicated of $\Pi_1, \ldots, \Pi_n$ on top of Σ would not always count as a proof (of θ from $\Delta_1 \cup \ldots \cup \Delta_n \cup \Gamma$).

But this apparent absence of unrestricted transitivity (for the system of core, hence *normal*, proof) imposes no limitation in principle. This perhaps surprising, but very welcome, result is secured by the following theorem.

Theorem 1 (Cut Elimination for Core Proof)
There is an effective method [,] *that transforms any two core proofs*

$$\begin{array}{cc} \Delta & A,\Gamma \\ \Pi & \Sigma \\ A & \theta \end{array} \qquad \textit{(where } A \notin \Gamma \textit{ and } \Gamma \textit{ may be empty)}$$

into a core proof $[\Pi\ \Sigma]$ of θ or of $\bot$ *from (some subset of)* $\Delta \cup \Gamma$.

Corollary 1 (Multiple Cut Elimination for Core Proof)
One can effectively transform the core proofs

$$\begin{array}{cccc} \Delta_1 & & \Delta_n & A_1,\ldots,A_n,\Gamma \\ \Pi_1 & ,\ldots, & \Pi_n & \Sigma \\ A_1 & & A_n & \theta \end{array} \qquad \textit{(where } A_1,\ldots,A_n \notin \Gamma \textit{ and } \Gamma \textit{ may be empty)}$$

into a core proof $[\Pi_1\ \ldots\ [[\Pi_n\ \Sigma]\ldots]$ of θ or of $\bot$ *from (some subset of)* $\Delta_1 \cup \ldots \Delta_n \cup \Gamma$.

Comment. In general the core proof $[\Pi_1\ \ldots\ [[\Pi_n\ \Sigma]\ldots]$ will depend on the order of the Π_i. So, for example, $[\Pi_1\ [\Pi_2\ \Sigma]]$ need not be identical to $[\Pi_2\ [\Pi_1\ \Sigma]]$.

Proof of Theorem 1. The operation [,] is defined inductively on the complexity of the proofs Π and Σ, and—where relevant—the complexity of the cut-sentence A. First we take care of the grounding cases, by means of the following four *grounding conversions.* Note that the operation applies to Π and to Σ (see case (4)) even if the conclusion of Π is not an undischarged assumption of Σ.

1. $[A\ \Sigma] = \Sigma$ (where A is an undischarged assumption of Σ).
2. $[\Pi\ A] = \Pi$ (where A is the conclusion of Π).
3. If no assumption-occurrence of A within Σ is the major premise of an elimination, but A is an undischarged assumption of Σ, then

$$[\Pi\ \Sigma] = \begin{array}{c} \Delta \\ \Pi \\ (A)\,,\Gamma \\ \Sigma \\ \theta \end{array}$$

4. If the conclusion of Π is $\perp$, then $[\Pi\ \Sigma] = \Pi$; otherwise, if the conclusion of Π is not an undischarged assumption of Σ, then $[\Pi\ \Sigma] = \Sigma$.

It remains only to consider cases where Π and Σ satisfy the following conditions.

(i) Π is a non-trivial proof of A (so we are not dealing with case (1) above),

(ii) A is an undischarged assumption of Σ (so we are not dealing with case (4) above), and

(iii) at least one assumption-occurrence of A in Σ is the major premise of an elimination (so we are not dealing with either case (2) or case (3) above).

Definition 1 *Cases satisfying conditions (i)-(iii) above will be called* ripe.

A ripe case is one where the arguments Π, Σ for the operation $[\Pi, \Sigma]$ can be represented graphically by the annotated proof-schemata

$$\begin{array}{cccc} & \Delta & & A\,,\Gamma \\ \textit{non-trivial} \longrightarrow & \Pi & \textit{occurs as MPE} \nearrow & \Sigma \\ & A & & \theta \end{array}$$

The rest of this discussion is devoted to the treatment of ripe cases. Ripe cases are of two kinds:

(a) the last step of Σ does *not* have the cut-sentence A as MPE;

(b) the last step of Σ *does* have the cut-sentence A as MPE.

Definition 2 *Ripe cases of type (a) are called* soft *ripe. Ripe cases of type (b) are called* hard *ripe.*

Soft ripe cases are in turn of two kinds:

(1) the last step of Σ is an introduction;

(2) the last step of Σ is an elimination.

Definition 3 *Cases of type (1) are called* soft ripe introductory*; cases of type (2) are called* soft ripe eliminative*.*

Hard ripe cases, likewise, are of two kinds:

(i) the last step of Π is a β-elimination, where $\beta \neq \neg$;

(ii) the last step of Π is an α-Introduction.

We now have the following exhaustive and non-overlapping classification of cases. This is important, since we need to deal with every possible case, and need also to deal with each case in a unique way. The dominant operator of the cut-sentence A is assumed to be α. We indicate in square brackets the kind of transformation (to be described below) that will apply to the cases in question.

(I) Grounding cases [Grounding conversions]

(II) Ripe cases
  (a) Soft ripe cases
    (1) Soft ripe introductory cases [I-Distribution conversions]
    (2) Soft ripe eliminative cases [E-Distribution conversions]
  (b) Hard ripe cases
    (i) Last step of Π is β-E, $\beta \neq \neg$ [Permutation conversions]
    (ii) Last step of Π is α-I [Reductions]

Definition 4 *Let us call any occurrence of A of the kind mentioned in (iii)—i.e., an assumption-occurrence of A in Σ that is the major premise of an elimination—an* MPE-occurrence of A *in* Σ*.*

Definition 5 *We say that a proof 'proves $\perp$' if its conclusion is $\perp$. (Of course, any such proof has some undischarged assumptions.)*

The operation $[\Pi\ \Sigma]$ distributes across terminal applications of rules of inference in Σ that do not have the cut-sentence A as MPE. (It is assumed, however, that A, which is the conclusion of the proof Π, is an undischarged assumption of the proof Σ.) The following *I- and E-distribution conversions* specify how to proceed under these circumstances.

If the proof Σ ends with an introduction (which of course involves no MPE) then we apply the appropriate I-distribution conversion, depending on the dominant operator in the conclusion of Σ. Let us illustrate with the case where the last step of Σ is $\wedge$-Introduction:

$$\left[\begin{array}{cc} \Delta & \Sigma_1 \quad \Sigma_2 \\ \Pi & \dfrac{\theta_1 \quad \theta_2}{\theta_1 \wedge \theta_2} \\ A & \end{array}\right] =_{df} \left[\,\Pi\ \ \Sigma_1\,\right] \ \text{ if } \left[\,\Pi\ \ \Sigma_1\,\right] \text{ proves } \perp\text{; otherwise,}$$

$$=_{df} \left[\,\Pi\ \ \Sigma_2\,\right] \ \text{ if } \left[\,\Pi\ \ \Sigma_2\,\right] \text{ proves } \perp\text{; otherwise,}$$

$$=_{df} \frac{\left[\,\Pi\ \ \Sigma_1\,\right]\ \left[\,\Pi\ \ \Sigma_2\,\right]}{\theta_1 \wedge \theta_2}$$

The same recipe can encoded graphically as follows:

$$\left[\begin{array}{cc} & \Sigma_1 \quad \Sigma_2 \\ \Pi & \dfrac{\theta_1 \quad \theta_2}{\theta_1 \wedge \theta_2} \end{array}\right] = \begin{array}{c} [\Pi\ \Sigma_1] \quad [\Pi\ \Sigma_2] \\ \theta_1/\perp \qquad \theta_2/\perp \\ \left\{ \dfrac{\qquad\qquad\qquad}{\theta_1 \wedge \theta_2} \right\} \end{array}$$

where the curly parentheses enclosing the final step of $\wedge$-I indicate that the step is not necessary if either $[\Pi\ \Sigma_1]$ or $[\Pi\ \Sigma_2]$ proves $\perp$. With that explanation, we can rewrite the right-hand side so that the $\wedge$-I distribution conversion reads

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$$\wedge\text{-I Distribution}\qquad \left[\;\Pi\;\; \dfrac{\begin{matrix}\Sigma_1 & \Sigma_2\\ \theta_1 & \theta_2\end{matrix}}{\theta_1\wedge\theta_2}\;\right] = \dfrac{\begin{matrix}[\Pi\ \Sigma_1] & [\Pi\ \Sigma_2]\\ \theta_1/\bot & \theta_2/\bot\end{matrix}}{\theta_1\wedge\theta_2/\bot}$$

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without any risk of miscontrual.

If the proof Σ ends with an elimination, whose MPE is not the cut-sentence A, then we apply the appropriate E-distribution conversion. Staying with conjunction for purposes of illustration, we have:

$$\left[\begin{array}{ll}\Delta & \qquad\Gamma\,,\overset{\text{—}(i)}{\varphi}\,,\overset{\text{—}(i)}{\psi}\\ \Pi & \qquad\quad\;\Theta\\ A & \dfrac{\varphi\wedge\psi\quad\theta}{\theta}\,(i)\end{array}\right] =_{df} \left[\,\Pi\ \ \Theta\,\right]$$

if $\left[\,\Pi\ \ \Theta\,\right]$ proves a subsequent of $\Gamma : \theta$;
otherwise,

$$=_{df}\ \dfrac{\varphi\wedge\psi\quad\left[\,\Pi\ \ \Theta\,\right]}{\bot}\quad \text{if } \left[\,\Pi\ \ \Theta\,\right] \text{ proves } \bot;$$

otherwise,

$$=_{df}\ \dfrac{\varphi\wedge\psi\quad\left[\,\Pi\ \ \Theta\,\right]}{\theta}$$

This can be encoded graphically as follows:

$$\left[\begin{array}{ll}\Delta & \qquad\Gamma\,,\overset{\text{—}(i)}{\varphi}\,,\overset{\text{—}(i)}{\psi}\\ \Pi & \qquad\quad\;\Theta\\ A & \dfrac{\varphi\wedge\psi\quad\theta}{\theta}\,(i)\end{array}\right] = \begin{array}{c}\dfrac{[\Pi\ \Theta]}{\theta/\bot}\\ \left\{\dfrac{\varphi\wedge\psi\qquad\downarrow}{\theta/\bot}\right\}\end{array}$$

where the curly parentheses indicate that the final step of $\wedge$-E is not necessary if $[\Pi\ \Theta]$ has neither φ nor ψ as an undischarged assumption. With that explanation, we can shorten the right-hand side even further and write

$$\land\text{-E Distribution}\qquad \left[\begin{array}{lc} \Delta & \Gamma\,,\overset{\text{—}(i)}{\varphi}\,,\overset{\text{—}(i)}{\psi} \\ \Pi & \Theta \\ A & \dfrac{\varphi\land\psi \quad \theta}{\theta}{\scriptstyle (i)} \end{array}\right] = \dfrac{\varphi\land\psi \quad \dfrac{[\Pi\ \Theta]}{\theta/\bot}}{\theta/\bot}$$

without any risk of misconstrual.

Note how the definition takes into account the possibility that the assumptions-to-be-discharged, namely φ and ψ, might both be absent from the resulting set of undischarged assumptions in the transformed proof $[\Pi\ \Theta]$. In that case there is no need for a terminal step of $\land$-E in the proof on the right. One simply takes the proof $[\Pi\ \Theta]$ for one's result. Similar remarks hold for $\lor$-E Distribution, $\rightarrow$-E Distribution, $\exists$-E Distribution, and $\forall$-E Distribution.

The consideration of epistemic gain applies at every stage of the execution of our effective procedure. If we ever come across a proof of a *proper subsequent* of the sequent 'to be proved', then we take that proof for our sought result. Otherwise, we build up the sought proof according to the recipe being explained. Our notations below must be read with this point in mind.

The remaining I- and E-distribution conversions for operators other than $\land$, using the same graphic abbreviation conventions, are as follows. Remember, it is being assumed that the conclusion A of the proof Π is *not* the MPE of the terminal step of the proof immediately to the right of Π (if that terminal step is an elimination), but *is* one of its undischarged assumptions.

$$\neg\text{-I Distribution}\qquad \left[\begin{array}{lc} \Delta & \Gamma\,,\overset{\text{—}(i)}{\varphi} \\ \Pi & \Sigma \\ A & \dfrac{\bot}{\neg\varphi}{\scriptstyle (i)} \end{array}\right] = \dfrac{\left[\begin{array}{lc} \Delta & \Gamma\,,\overset{\text{—}(i)}{\varphi} \\ \Pi & \Sigma \\ A & \bot \end{array}\right]}{\dfrac{\bot}{\neg\varphi}{\scriptstyle (i)}}$$

¬-E Distribution

$$\left[\begin{array}{ll} \Delta & \Gamma \\ \Pi & \Sigma \\ A & \dfrac{\neg\varphi \quad \varphi}{\bot} \end{array}\right] = \dfrac{\neg\varphi \qquad \dfrac{\left[\begin{array}{ll} \Delta & \Gamma \\ \Pi & \Sigma \\ A & \varphi \end{array}\right]}{\varphi/\bot}}{\bot}$$

∨-I Distribution

$$\left[\begin{array}{ll} \Delta & \Gamma \\ \Pi & \Sigma \\ A & \dfrac{\varphi_i}{\varphi_1 \vee \varphi_2} \end{array}\right] = \dfrac{\left[\begin{array}{ll} \Delta & \Gamma \\ \Pi & \Sigma \\ A & \varphi_i \end{array}\right]}{\dfrac{\varphi_i/\bot}{\varphi_1 \vee \varphi_2}}$$

∨-E Distribution

$$\left[\begin{array}{ll} \Delta & \\ \Pi & \\ A & \dfrac{\varphi_1 \vee \varphi_2 \qquad \begin{array}{c} \overset{(i)}{\overline{\varphi_1}}\,, \Gamma_1 \\ \Sigma_1 \\ \theta/\bot \end{array} \qquad \begin{array}{c} \overset{(i)}{\overline{\varphi_2}}\,, \Gamma_2 \\ \Sigma_2 \\ \theta/\bot \end{array}}{\theta/\bot}\,(i) \end{array}\right]$$

$$= \dfrac{\varphi_1 \vee \varphi_2 \qquad \dfrac{\left[\begin{array}{ll} \Delta & \overset{(i)}{\overline{\varphi_1}}\,, \Gamma_1 \\ \Pi & \Sigma_1 \\ A & \theta/\bot \end{array}\right]}{\theta/\bot} \qquad \dfrac{\left[\begin{array}{ll} \Delta & \overset{(i)}{\overline{\varphi_2}}\,, \Gamma_2 \\ \Pi & \Sigma_2 \\ A & \theta/\bot \end{array}\right]}{\theta/\bot}}{\theta/\bot}\,(i)$$

→-I Distribution

$$\left[\begin{array}{ll} \Delta & \Gamma,\ \overline{\varphi}^{(i)} \\ \Pi & \Sigma \\ A & \dfrac{\bot}{\varphi \to \psi}\,(i) \end{array}\right] = \dfrac{\left[\begin{array}{ll} \Delta & \Gamma,\ \overline{\varphi}^{(i)} \\ \Pi & \Sigma \\ A & \bot \end{array}\right]}{\dfrac{\bot}{\varphi \to \psi}\,(i)}$$

→-I Distribution

$$\left[\begin{array}{cc} \Delta & \Gamma\,,\ \overline{\varphi}^{\,(i)} \\ \Pi & \Sigma \\ A & \dfrac{\psi}{\varphi \to \psi}{\scriptstyle(i)} \end{array}\right] = \dfrac{\dfrac{\left[\begin{array}{cc} \Delta & \Gamma\,,\ \overline{\varphi}^{\,(i)} \\ \Pi & \Sigma \\ A & \psi \end{array}\right]}{\psi/\bot}}{\varphi \to \psi}{\scriptstyle(i)}$$

→-E Distribution

$$\left[\begin{array}{ccccl} \Delta & & \Gamma_1 & \Gamma_2\,,\ \overline{\psi}^{\,(1)} & \\ \Pi & & \Sigma_1 & \Sigma_2 & \\ A & \varphi\to\psi & \varphi & \theta & {\scriptstyle(1)} \\ \cline{2-4} & \multicolumn{3}{c}{\theta} & \end{array}\right]$$

$$= \dfrac{\varphi\to\psi \qquad \dfrac{\left[\begin{array}{cc} \Delta & \Gamma_1 \\ \Pi & \Sigma_1 \\ A & \varphi \end{array}\right]}{\varphi/\bot} \qquad \dfrac{\left[\begin{array}{cc} \Delta & \Gamma_2\,,\ \overline{\psi}^{\,(1)} \\ \Pi & \Sigma_2 \\ A & \theta \end{array}\right]}{\theta/\bot}}{\theta/\bot}{\scriptstyle(1)}$$

(Note that if the minor subproof on the right proves $\bot$, then the minor proof is taken for the whole transform.)

∃-I Distribution

$$\left[\begin{array}{cc} \begin{matrix} \Delta \\ \Pi \\ A \end{matrix} & \dfrac{\begin{matrix} \Gamma \\ \Sigma \\ \varphi^x_t \end{matrix}}{\exists x \varphi} \end{array}\right] = \dfrac{\dfrac{\left[\begin{array}{cc} \Delta & \Gamma \\ \Pi & \Sigma \\ A & \varphi^x_t \end{array}\right]}{\varphi^x_t/\bot}}{\exists x \varphi}$$

∃-E Distribution

$$\left[\begin{array}{cc}\Delta & \\ \Pi & \\ A & \dfrac{\exists x\varphi \qquad \begin{array}{c}\Gamma\,,\,\overline{\varphi^x_a}^{\,(i)}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle (i)}\end{array}\right] = \dfrac{\exists x\varphi \qquad \dfrac{\left[\begin{array}{cc}\Delta & \Gamma\,,\,\overline{\varphi^x_a}^{\,(i)}\\ \Pi & \Sigma\\ A & \theta\end{array}\right]}{\theta/\bot}}{\theta/\bot}{\scriptstyle (i)}$$

∀-I Distribution

$$\left[\begin{array}{cc}\Delta & \Gamma\\ \Pi & \Sigma\\ A & \dfrac{\varphi}{\forall x\varphi^a_x}\end{array}\right] = \dfrac{\dfrac{\left[\begin{array}{cc}\Delta & \Gamma\\ \Pi & \Sigma\\ A & \varphi\end{array}\right]}{\varphi/\bot}}{\forall x\varphi^a_x}$$

∀-E Distribution

$$\left[\begin{array}{cc}\Delta & \\ \Pi & \\ A & \dfrac{\forall x\varphi \qquad \begin{array}{c}\Gamma,\,\overline{\varphi^x_{t_1}}^{\,(i)},\ldots,\overline{\varphi^x_{t_n}}^{\,(i)}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle (i)}\end{array}\right]$$

$$= \dfrac{\forall x\varphi \qquad \dfrac{\left[\begin{array}{cc}\Delta & \Gamma,\,\overline{\varphi^x_{t_1}}^{\,(i)},\ldots,\overline{\varphi^x_{t_n}}^{\,(i)}\\ \Pi & \Sigma\\ A & \theta\end{array}\right]}{\theta/\bot}}{\theta/\bot}{\scriptstyle (i)}$$

Our distribution conversions above tell one exactly how to compose $[\Pi\ \Sigma]$ in soft ripe cases from (suitable transforms of) subproofs of Π and of Σ. These conversions handle cases where

1. the final step of Σ is an introduction, or
2. the final step of Σ is an elimination, but does not have the conclusion of Π (= the cut-sentence A) as its major premise.

We now need to address only the question of how to effect these transformations

$$\left[\begin{array}{cc} \Delta & A,\Gamma \\ \Pi & \Sigma \\ A & \theta \end{array}\right]$$

in hard ripe cases, that is, when the cut-sentence A occurs as an undischarged assumption of Σ, and that occurrence is as the major premise of the final step of Σ, which is an elimination.

We proceed by cases, determined by

1. the dominant operator α of A (thereby determining that α-E is the E-rule applied at the final step in Σ);
2. the rule applied at the final step of Π (which can only be an elimination other than $\neg$-E, or an application of α-I).[4]

There are thirty six cases to consider, and they are laid out systematically in the table below. Each entry $\beta\alpha$ (above the horizontal line) represents the case where the last step of Π is an application of β-E, and the last step of Σ is an application of α-E with major premise A (which of course leaves open the possibility that A occurs elsewhere as an undischarged assumption of Σ).

Entries of the form $\alpha\alpha$ of course represent steps where the two MPEs involved will not in general be the same. The final row of the table (below the horizontal line) is to be understood as involving cases where the last step of Π is an application of α-I (as indicated by the subscript I), and the last step of Σ is an application of α-E.

$\wedge\neg$	$\wedge\wedge$	$\wedge\vee$	$\wedge\rightarrow$	$\wedge\exists$	$\wedge\forall$
$\vee\neg$	$\vee\wedge$	$\vee\vee$	$\vee\rightarrow$	$\vee\exists$	$\vee\forall$
$\rightarrow\neg$	$\rightarrow\wedge$	$\rightarrow\vee$	$\rightarrow\rightarrow$	$\rightarrow\exists$	$\rightarrow\forall$
$\exists\neg$	$\exists\wedge$	$\exists\vee$	$\exists\rightarrow$	$\exists\exists$	$\exists\forall$
$\forall\neg$	$\forall\wedge$	$\forall\vee$	$\forall\rightarrow$	$\forall\exists$	$\forall\forall$

---

[4] The final step of Π cannot be an application of $\neg$-E, since its conclusion is $\bot$, which cannot feature as an undischarged assumption of *any* proof, hence not of Σ.

$\neg_I\neg$	$\wedge_I\wedge$	$\vee_I\vee$	$\rightarrow_I\rightarrow$	$\exists_I\exists$	$\forall_I\forall$

We shall work our way down each column, from the left to the right, leaving the final row for consideration at the end. *All cases above the horizontal line involve what are known as 'permutation conversions'.* With a permutation conversion, the basic aim is to get the terminal elimination of Π to be terminal in the transform [Π Σ]. In the schemata below, the cut-sentence is (or its immediate subsentences are) in roman. The terminal MPE of Π is in Greek font. This helps the reader to track the effect of the conversion or reduction in question. The starred parameters for ∃-E in the transforms for cases of the form $\exists\alpha$ are to be chosen so as to ensure that the application in question of ∃-E is formally correct. By inspection, this can always be done.

Note that the effect of every permutation conversion below is to reduce the complexity of the proofs Π and Σ to which the operation [,] needs to be applied.

The transformation steps for the cases corresponding to entries below the horizontal line in the table above are known as 'reductions'. The effect of a reduction is to reduce the complexity of the cut-sentence with respect to which execution of the operation [,] is still called for.

The list of permutation conversion now follows. Thereafter, we give the list of reductions. We lapse into English only to remark when the one list ends and the other begins.

As with the distribution conversions, the transforms produced by the permutation conversions can be more economical than what is shown in full on the right. This can happen when the transform of a subproof establishes a strong enough result: either it proves ⊥, or it makes do without using any of the assumptions that would otherwise have to be discharged by the terminal elimination lower down.

$$\wedge\neg \quad \left[\begin{array}{cc} \dfrac{\varphi\wedge\psi \quad \begin{array}{c}\underbrace{\Xi\ \ \overline{\varphi}^{(2)}\ \ \overline{\psi}^{(2)}}\\ \Theta\\ \neg A\end{array}}{\neg A}(2) & \begin{array}{c}\Delta\\ \Pi\\ \dfrac{\neg A \quad A}{\bot}(1)\end{array}\end{array}\right] = \dfrac{\varphi\wedge\psi \qquad \dfrac{\left[\begin{array}{cc}\begin{array}{c}\underbrace{\Xi\ \ \overline{\varphi}^{(2)}\ \ \overline{\psi}^{(2)}}\\ \Theta\\ \neg A\end{array} & \begin{array}{c}\Delta\\ \Pi\\ \dfrac{\neg A \quad A}{\bot}(1)\end{array}\end{array}\right]}{\bot}}{\bot}(2)$$

$$\vee\neg \quad \left[\begin{array}{cc} \dfrac{\varphi\vee\psi \quad \begin{array}{c}\overline{\varphi}^{(2)}\\ \Theta_1\\ \neg A\end{array} \quad \begin{array}{c}\overline{\psi}^{(2)}\\ \Theta_2\\ \neg A\end{array}}{\neg A}(2) & \begin{array}{c}\Delta\\ \Pi\\ \dfrac{\neg A \quad A}{\bot}(1)\end{array}\end{array}\right] = \dfrac{\varphi\vee\psi \qquad \dfrac{\left[\begin{array}{cc}\begin{array}{c}\overline{\varphi}^{(2)}\\ \Theta_1\\ \neg A\end{array} & \begin{array}{c}\Delta\\ \Pi\\ \dfrac{\neg A \quad A}{\bot}(1)\end{array}\end{array}\right]}{\bot} \quad \dfrac{\left[\begin{array}{cc}\begin{array}{c}\overline{\psi}^{(2)}\\ \Theta_2\\ \neg A\end{array} & \begin{array}{c}\Delta\\ \Pi\\ \dfrac{\neg A \quad A}{\bot}(1)\end{array}\end{array}\right]}{\bot}}{\bot}(2)$$

$$\to\neg \quad \left[\begin{array}{cc} \dfrac{\varphi\to\psi \quad \begin{array}{c}\Xi_1\\ \Theta_1\\ \varphi\end{array} \quad \begin{array}{c}\Xi_2, \overline{\psi}^{(2)}\\ \Theta_2\\ \neg A\end{array}}{\neg A}(2) & \begin{array}{c}\Delta\\ \Pi\\ \dfrac{\neg A \quad A}{\bot}(1)\end{array}\end{array}\right] = \dfrac{\varphi\to\psi \quad \begin{array}{c}\Xi_1\\ \Theta_1\\ \varphi\end{array} \qquad \dfrac{\left[\begin{array}{cc}\begin{array}{c}\Xi_2, \overline{\psi}^{(2)}\\ \Theta_2\\ \neg A\end{array} & \begin{array}{c}\Delta\\ \Pi\\ \dfrac{\neg A \quad A}{\bot}(1)\end{array}\end{array}\right]}{\bot}}{\bot}(2)$$

$$\exists\neg \quad \left[\begin{array}{cc} \dfrac{\exists x\varphi \quad \begin{array}{c}\Xi, \overline{\varphi^x_a}^{(2)}\\ \Theta\\ \neg A\end{array}}{\neg A}(2) & \begin{array}{c}\Delta\\ \Pi\\ \dfrac{\neg A \quad A}{\bot}(1)\end{array}\end{array}\right] = \dfrac{\exists x\varphi \qquad \dfrac{\left[\begin{array}{cc}\begin{array}{c}\Xi, \overline{\varphi^x_{a^*}}^{(2)}\\ \Theta\\ \neg A\end{array} & \begin{array}{c}\Delta\\ \Pi\\ \dfrac{\neg A \quad A}{\bot}(1)\end{array}\end{array}\right]}{\bot}}{\bot}(2)$$

$$\forall\neg \qquad \left[\begin{array}{cc} \dfrac{\forall x\varphi \qquad \begin{array}{c}\Xi, \overline{\varphi^x_{t_1}}^{(2)}, \ldots, \overline{\varphi^x_{t_n}}^{(2)}\\ \Theta \\ \neg A\end{array}}{\neg A}{\scriptstyle (2)} & \begin{array}{c}\Delta\\ \Pi\\ \dfrac{\neg A \quad A}{\bot}{\scriptstyle (1)}\end{array}\end{array}\right] = \dfrac{\forall x\varphi \qquad \dfrac{\left[\begin{array}{cc}\begin{array}{c}\Xi, \overline{\varphi^x_{t_1}}^{(2)}, \ldots, \overline{\varphi^x_{t_n}}^{(2)}\\ \Theta\\ \neg A\end{array} & \begin{array}{c}\Delta\\ \Pi\\ \dfrac{\neg A \quad A}{\bot}{\scriptstyle (1)}\end{array}\end{array}\right]}{\bot}}{\bot}{\scriptstyle (2)}$$

Note that if the proof on the right ends with the step of ∀-E indicated, it will be because some, but not necessarily all, of the assumptions $\varphi^x_{t_1}, \ldots, \varphi^x_{t_n}$ remain undischarged within the embedded transform. A similar remark applies to all cases of permutative conversions of the form $\forall\alpha$.

$$\wedge\wedge \qquad \left[\begin{array}{cc} \dfrac{\varphi\wedge\psi \qquad \begin{array}{c}\underbrace{\Xi \quad \overline{\varphi}^{(2)} \quad \overline{\psi}^{(2)}}_{\textstyle\Theta}\\ A\wedge B\end{array}}{A\wedge B}{\scriptstyle (2)} & \begin{array}{c}\Gamma, \overline{A}^{(1)}, \overline{B}^{(1)}\\ \Sigma\\ \dfrac{A\wedge B \qquad \theta}{\theta}{\scriptstyle (1)}\end{array}\end{array}\right] = \dfrac{\varphi\wedge\psi \qquad \dfrac{\left[\begin{array}{cc}\begin{array}{c}\underbrace{\Xi \quad \overline{\varphi}^{(2)} \quad \overline{\psi}^{(2)}}_{\textstyle\Theta}\\ A\wedge B\end{array} & \begin{array}{c}\Gamma, \overline{A}^{(1)}, \overline{B}^{(1)}\\ \Sigma\\ \dfrac{A\wedge B \qquad \theta}{\theta}{\scriptstyle (1)}\end{array}\end{array}\right]}{\theta/\bot}}{\theta/\bot}{\scriptstyle (2)}$$

$$\vee\wedge \qquad \left[\begin{array}{cc} \dfrac{\varphi\vee\psi \qquad \begin{array}{c}\overline{\varphi}^{(2)}\\ \Theta_1\\ A\wedge B\end{array} \qquad \begin{array}{c}\overline{\psi}^{(2)}\\ \Theta_2\\ A\wedge B\end{array}}{A\wedge B}{\scriptstyle (2)} & \begin{array}{c}\Gamma, \overline{A}^{(1)}, \overline{B}^{(1)}\\ \Sigma\\ \dfrac{A\wedge B \qquad \theta}{\theta}{\scriptstyle (1)}\end{array}\end{array}\right]$$

$$= \dfrac{\varphi\vee\psi \qquad \dfrac{\left[\begin{array}{cc}\begin{array}{c}\overline{\varphi}^{(2)}\\ \Theta_1\\ A\wedge B\end{array} & \begin{array}{c}\Gamma, \overline{A}^{(1)}, \overline{B}^{(1)}\\ \Sigma\\ \dfrac{A\wedge B \qquad \theta}{\theta}{\scriptstyle (1)}\end{array}\end{array}\right]}{\theta/\bot} \qquad \dfrac{\left[\begin{array}{cc}\begin{array}{c}\overline{\psi}^{(2)}\\ \Theta_2\\ A\wedge B\end{array} & \begin{array}{c}\Gamma, \overline{A}^{(1)}, \overline{B}^{(1)}\\ \Sigma\\ \dfrac{A\wedge B \qquad \theta}{\theta}{\scriptstyle (1)}\end{array}\end{array}\right]}{\theta/\bot}}{\theta/\bot}{\scriptstyle (2)}$$

$$\to\wedge \qquad \left[\dfrac{\varphi\to\psi \quad \begin{matrix}\Xi_1\\ \Theta_1\\ \varphi\end{matrix} \quad \begin{matrix}\Xi_2,\overline{\psi}^{(2)}\\ \Theta_2\\ A{\wedge}B\end{matrix}}{A\wedge B}{\scriptstyle(2)} \qquad \dfrac{A{\wedge}B \quad \begin{matrix}\Gamma,\overline{A}^{(1)},\overline{B}^{(1)}\\ \Sigma\\ \theta\end{matrix}}{\theta}{\scriptstyle(1)}\right] = \dfrac{\varphi\to\psi \quad \begin{matrix}\Xi_1\\ \Theta_1\\ \varphi\end{matrix} \quad \dfrac{\left[\begin{matrix}\Xi_2,\overline{\psi}^{(2)}\\ \Theta_2\\ A{\wedge}B\end{matrix} \quad \dfrac{A{\wedge}B \quad \begin{matrix}\Gamma,\overline{A}^{(1)},\overline{B}^{(1)}\\ \Sigma\\ \theta\end{matrix}}{\theta}{\scriptstyle(1)}\right]}{\theta/\bot}}{\theta/\bot}{\scriptstyle(2)}$$

$$\exists\wedge \qquad \left[\dfrac{\exists x\varphi \quad \begin{matrix}\Xi,\overline{\varphi^x_a}^{(2)}\\ \Theta\\ A{\wedge}B\end{matrix}}{A\wedge B}{\scriptstyle(2)} \qquad \dfrac{A{\wedge}B \quad \begin{matrix}\Gamma,\overline{A}^{(1)},\overline{B}^{(1)}\\ \Sigma\\ \theta\end{matrix}}{\theta}{\scriptstyle(1)}\right] = \dfrac{\exists x\varphi \quad \dfrac{\left[\begin{matrix}\Xi,\overline{\varphi^x_{a^*}}^{(2)}\\ \Theta\\ A{\wedge}B\end{matrix} \quad \dfrac{A{\wedge}B \quad \begin{matrix}\Gamma,\overline{A}^{(1)},\overline{B}^{(1)}\\ \Sigma\\ \theta\end{matrix}}{\theta}{\scriptstyle(1)}\right]}{\theta/\bot}}{\theta/\bot}{\scriptstyle(2)}$$

$$\forall\wedge \qquad \left[\dfrac{\forall x\varphi \quad \begin{matrix}\Xi,\overline{\varphi^x_{t_1}}^{(2)},\ldots,\overline{\varphi^x_{t_n}}^{(2)}\\ \Theta\\ A{\wedge}B\end{matrix}}{A\wedge B}{\scriptstyle(2)} \qquad \dfrac{A{\wedge}B \quad \begin{matrix}\Gamma,\overline{A}^{(1)},\overline{B}^{(1)}\\ \Sigma\\ \theta\end{matrix}}{\theta}{\scriptstyle(1)}\right] = \dfrac{\forall x\varphi \quad \dfrac{\left[\begin{matrix}\Xi,\overline{\varphi^x_{t_1}}^{(2)},\ldots,\overline{\varphi^x_{t_n}}^{(2)}\\ \Theta\\ A{\wedge}B\end{matrix} \quad \dfrac{A{\wedge}B \quad \begin{matrix}\Gamma,\overline{A}^{(1)},\overline{B}^{(1)}\\ \Sigma\\ \theta\end{matrix}}{\theta}{\scriptstyle(1)}\right]}{\theta/\bot}}{\theta/\bot}{\scriptstyle(2)}$$

$$\wedge\vee \qquad \left[\dfrac{\varphi\wedge\psi \quad \begin{matrix}\underbrace{\Xi\ \overline{\varphi}^{(2)}\ \ \overline{\psi}^{(2)}}\\ \Theta\\ A{\vee}B\end{matrix}}{A\vee B}{\scriptstyle(2)} \qquad \dfrac{A{\vee}B \quad \begin{matrix}\Delta,\overline{A}^{(1)}\\ \Pi\\ \theta\end{matrix} \quad \begin{matrix}\Gamma,\overline{B}^{(1)}\\ \Sigma\\ \theta\end{matrix}}{\theta}{\scriptstyle(1)}\right] = \dfrac{\varphi\wedge\psi \quad \dfrac{\left[\begin{matrix}\underbrace{\Xi\ \overline{\varphi}^{(2)}\ \ \overline{\psi}^{(2)}}\\ \Theta\\ A{\vee}B\end{matrix} \quad \dfrac{A{\vee}B \quad \begin{matrix}\Delta,\overline{A}^{(1)}\\ \Pi\\ \theta\end{matrix} \quad \begin{matrix}\Gamma,\overline{B}^{(1)}\\ \Sigma\\ \theta\end{matrix}}{\theta}{\scriptstyle(1)}\right]}{\theta/\bot}}{\theta/\bot}{\scriptstyle(2)}$$

∧∨

$$\left[\ \dfrac{\varphi\wedge\psi \qquad \begin{array}{c}\underbrace{\Xi\ \ \overline{\varphi}^{(2)}\ \ \overline{\psi}^{(2)}}\\ \Theta\\ A\vee B\end{array}}{A\vee B}{\scriptstyle(2)} \qquad \dfrac{A\vee B \quad \begin{array}{c}\Delta,\overline{A}^{(1)}\\ \Pi\\ \theta\end{array} \quad \begin{array}{c}\Gamma,\overline{B}^{(1)}\\ \Sigma\\ \bot\end{array}}{\theta}{\scriptstyle(1)}\ \right] = \dfrac{\varphi\wedge\psi \qquad \dfrac{\left[\ \begin{array}{c}\underbrace{\Xi\ \ \overline{\varphi}^{(2)}\ \ \overline{\psi}^{(2)}}\\ \Theta\\ A\vee B\end{array} \quad \dfrac{A\vee B \quad \begin{array}{c}\Delta,\overline{A}^{(1)}\\ \Pi\\ \theta\end{array} \quad \begin{array}{c}\Gamma,\overline{B}^{(1)}\\ \Sigma\\ \bot\end{array}}{\theta}{\scriptstyle(1)}\ \right]}{\theta/\bot}}{\theta/\bot}{\scriptstyle(2)}$$

∧∨

$$\left[\ \dfrac{\varphi\wedge\psi \qquad \begin{array}{c}\underbrace{\Xi\ \ \overline{\varphi}^{(2)}\ \ \overline{\psi}^{(2)}}\\ \Theta\\ A\vee B\end{array}}{A\vee B}{\scriptstyle(2)} \qquad \dfrac{A\vee B \quad \begin{array}{c}\Delta,\overline{A}^{(1)}\\ \Pi\\ \bot\end{array} \quad \begin{array}{c}\Gamma,\overline{B}^{(1)}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle(1)}\ \right] = \dfrac{\varphi\wedge\psi \qquad \dfrac{\left[\ \begin{array}{c}\underbrace{\Xi\ \ \overline{\varphi}^{(2)}\ \ \overline{\psi}^{(2)}}\\ \Theta\\ A\vee B\end{array} \quad \dfrac{A\vee B \quad \begin{array}{c}\Delta,\overline{A}^{(1)}\\ \Pi\\ \bot\end{array} \quad \begin{array}{c}\Gamma,\overline{B}^{(1)}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle(1)}\ \right]}{\theta/\bot}}{\theta/\bot}{\scriptstyle(2)}$$

∨∨

$$\left[\ \dfrac{\varphi\vee\psi \qquad \begin{array}{c}\Xi_1,\overline{\varphi}^{(2)}\\ \Theta_1\\ A\vee B\end{array} \qquad \begin{array}{c}\Xi_2,\overline{\psi}^{(2)}\\ \Theta_2\\ A\vee B\end{array}}{A\vee B}{\scriptstyle(2)} \qquad \dfrac{A\vee B \quad \begin{array}{c}\Delta,\overline{A}^{(1)}\\ \Pi\\ \theta\end{array} \quad \begin{array}{c}\Gamma,\overline{B}^{(1)}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle(1)}\ \right]$$

$$= \dfrac{\varphi\vee\psi \qquad \dfrac{\left[\ \begin{array}{c}\Xi_1,\overline{\varphi}^{(2)}\\ \Theta_1\\ A\vee B\end{array} \quad \dfrac{A\vee B \quad \begin{array}{c}\Delta,\overline{A}^{(1)}\\ \Pi\\ \theta\end{array} \quad \begin{array}{c}\Gamma,\overline{B}^{(1)}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle(1)}\ \right]}{\theta/\bot} \qquad \dfrac{\left[\ \begin{array}{c}\Xi_2,\overline{\psi}^{(2)}\\ \Theta_2\\ A\vee B\end{array} \quad \dfrac{A\vee B \quad \begin{array}{c}\Delta,\overline{A}^{(1)}\\ \Pi\\ \theta\end{array} \quad \begin{array}{c}\Gamma,\overline{B}^{(1)}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle(1)}\ \right]}{\theta/\bot}}{\theta/\bot}{\scriptstyle(2)}$$

$$\vee\vee \quad \left[\dfrac{\varphi\vee\psi \quad \begin{matrix}\Xi_1, \overline{\varphi}^{(2)}\\ \Theta_1 \\ A\vee B\end{matrix} \quad \begin{matrix}\Xi_2, \overline{\psi}^{(2)}\\ \Theta_2\\ A\vee B\end{matrix}}{A\vee B}\,{\scriptstyle (2)} \quad \dfrac{A\vee B \quad \begin{matrix}\Delta, \overline{A}^{(1)}\\ \Pi\\ \theta\end{matrix} \quad \begin{matrix}\Gamma\,, \overline{B}^{(1)}\\ \Sigma\\ \bot\end{matrix}}{\theta}\,{\scriptstyle (1)} \right]$$

$$= \dfrac{\varphi\vee\psi \quad \dfrac{\left[\begin{matrix}\Xi_1, \overline{\varphi}^{(2)}\\ \Theta_1\\ A\vee B\end{matrix} \quad \dfrac{A\vee B \quad \begin{matrix}\Delta, \overline{A}^{(1)}\\ \Pi\\ \theta\end{matrix} \quad \begin{matrix}\Gamma\,, \overline{B}^{(1)}\\ \Sigma\\ \bot\end{matrix}}{\theta}\,{\scriptstyle (1)}\right]}{\theta/\bot} \quad \dfrac{\left[\begin{matrix}\Xi_2, \overline{\psi}^{(2)}\\ \Theta_2\\ A\vee B\end{matrix} \quad \dfrac{A\vee B \quad \begin{matrix}\Delta, \overline{A}^{(1)}\\ \Pi\\ \theta\end{matrix} \quad \begin{matrix}\Gamma\,, \overline{B}^{(1)}\\ \Sigma\\ \bot\end{matrix}}{\theta}\,{\scriptstyle (1)}\right]}{\theta/\bot}}{\theta/\bot}\,{\scriptstyle (2)}$$

$$\vee\vee \quad \left[\dfrac{\varphi\vee\psi \quad \begin{matrix}\Xi_1, \overline{\varphi}^{(2)}\\ \Theta_1 \\ A\vee B\end{matrix} \quad \begin{matrix}\Xi_2, \overline{\psi}^{(2)}\\ \Theta_2\\ A\vee B\end{matrix}}{A\vee B}\,{\scriptstyle (2)} \quad \dfrac{A\vee B \quad \begin{matrix}\Delta, \overline{A}^{(1)}\\ \Pi\\ \bot\end{matrix} \quad \begin{matrix}\Gamma\,, \overline{B}^{(1)}\\ \Sigma\\ \theta\end{matrix}}{\theta}\,{\scriptstyle (1)} \right]$$

$$= \dfrac{\varphi\vee\psi \quad \dfrac{\left[\begin{matrix}\Xi_1, \overline{\varphi}^{(2)}\\ \Theta_1\\ A\vee B\end{matrix} \quad \dfrac{A\vee B \quad \begin{matrix}\Delta, \overline{A}^{(1)}\\ \Pi\\ \bot\end{matrix} \quad \begin{matrix}\Gamma\,, \overline{B}^{(1)}\\ \Sigma\\ \theta\end{matrix}}{\theta}\,{\scriptstyle (1)}\right]}{\theta/\bot} \quad \dfrac{\left[\begin{matrix}\Xi_2, \overline{\psi}^{(2)}\\ \Theta_2\\ A\vee B\end{matrix} \quad \dfrac{A\vee B \quad \begin{matrix}\Delta, \overline{A}^{(1)}\\ \Pi\\ \bot\end{matrix} \quad \begin{matrix}\Gamma\,, \overline{B}^{(1)}\\ \Sigma\\ \theta\end{matrix}}{\theta}\,{\scriptstyle (1)}\right]}{\theta/\bot}}{\theta/\bot}\,{\scriptstyle (2)}$$

$$\to\vee \quad \left[\dfrac{\varphi\to\psi \quad \begin{matrix}\Xi_1\\ \Theta_1 \\ \varphi\end{matrix} \quad \begin{matrix}\Xi_2, \overline{\psi}^{(2)}\\ \Theta_2\\ A\vee B\end{matrix}}{A\vee B}\,{\scriptstyle (2)} \quad \dfrac{A\vee B \quad \begin{matrix}\Delta, \overline{A}^{(1)}\\ \Pi\\ \theta\end{matrix} \quad \begin{matrix}\Gamma\,, \overline{B}^{(1)}\\ \Sigma\\ \theta\end{matrix}}{\theta}\,{\scriptstyle (1)} \right]$$

$$= \qquad \cfrac{\varphi \to \psi \qquad \begin{matrix}\Xi_1\\ \Theta_1\\ \varphi\end{matrix} \qquad \cfrac{\left[\begin{matrix}\Xi_2, \overline{\psi}^{(2)}\\ \Theta_2\\ A\vee B\end{matrix} \qquad \cfrac{A\vee B \quad \begin{matrix}\Delta, \overline{A}^{(1)}\\ \Pi\\ \theta\end{matrix} \quad \begin{matrix}\Gamma\,, \overline{B}^{(1)}\\ \Sigma\\ \theta\end{matrix}}{\theta}{\scriptstyle(1)}\right]}{\theta/\bot}}{\theta/\bot}{\scriptstyle(2)}$$

$$\to\vee \quad \left[\cfrac{\varphi \to \psi \quad \begin{matrix}\Xi_1\\ \Theta_1\\ \varphi\end{matrix} \quad \begin{matrix}\Xi_2, \overline{\psi}^{(2)}\\ \Theta_2\\ A\vee B\end{matrix}}{A \vee B}{\scriptstyle(2)} \qquad \cfrac{A\vee B \quad \begin{matrix}\Delta, \overline{A}^{(1)}\\ \Pi\\ \theta\end{matrix} \quad \begin{matrix}\Gamma\,, \overline{B}^{(1)}\\ \Sigma\\ \bot\end{matrix}}{\theta}{\scriptstyle(1)}\right]$$

$$= \qquad \cfrac{\varphi \to \psi \qquad \begin{matrix}\Xi_1\\ \Theta_1\\ \varphi\end{matrix} \qquad \cfrac{\left[\begin{matrix}\Xi_2, \overline{\psi}^{(2)}\\ \Theta_2\\ A\vee B\end{matrix} \qquad \cfrac{A\vee B \quad \begin{matrix}\Delta, \overline{A}^{(1)}\\ \Pi\\ \theta\end{matrix} \quad \begin{matrix}\Gamma\,, \overline{B}^{(1)}\\ \Sigma\\ \bot\end{matrix}}{\theta}{\scriptstyle(1)}\right]}{\theta/\bot}}{\theta/\bot}{\scriptstyle(2)}$$

$$\to\vee \quad \left[\cfrac{\varphi \to \psi \quad \begin{matrix}\Xi_1\\ \Theta_1\\ \varphi\end{matrix} \quad \begin{matrix}\Xi_2, \overline{\psi}^{(2)}\\ \Theta_2\\ A\vee B\end{matrix}}{A \vee B}{\scriptstyle(2)} \qquad \cfrac{A\vee B \quad \begin{matrix}\Delta, \overline{A}^{(1)}\\ \Pi\\ \bot\end{matrix} \quad \begin{matrix}\Gamma\,, \overline{B}^{(1)}\\ \Sigma\\ \theta\end{matrix}}{\theta}{\scriptstyle(1)}\right]$$

$$= \qquad \cfrac{\varphi \to \psi \qquad \begin{matrix}\Xi_1\\ \Theta_1\\ \varphi\end{matrix} \qquad \cfrac{\left[\begin{matrix}\Xi_2, \overline{\psi}^{(2)}\\ \Theta_2\\ A\vee B\end{matrix} \qquad \cfrac{A\vee B \quad \begin{matrix}\Delta, \overline{A}^{(1)}\\ \Pi\\ \bot\end{matrix} \quad \begin{matrix}\Gamma\,, \overline{B}^{(1)}\\ \Sigma\\ \theta\end{matrix}}{\theta}{\scriptstyle(1)}\right]}{\theta/\bot}}{\theta/\bot}{\scriptstyle(2)}$$

$$\exists\vee\qquad \left[\;\dfrac{\exists x\varphi \qquad \begin{array}{c}\Xi,\overline{\varphi^x_a}^{(2)}\\ \Theta\\ A\vee B\end{array}}{A\vee B}{\scriptstyle(2)} \qquad \dfrac{A\vee B \quad \begin{array}{c}\Delta,\overline{A}^{(1)}\\ \Pi\\ \theta\end{array} \quad \begin{array}{c}\Gamma\,,\overline{B}^{(1)}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle(1)}\;\right] \;=\; \dfrac{\exists x\varphi \qquad \dfrac{\left[\;\begin{array}{c}\Xi,\overline{\varphi^x_{a^*}}^{(2)}\\ \Theta\\ A\vee B\end{array} \quad \dfrac{A\vee B \quad \begin{array}{c}\Delta,\overline{A}^{(1)}\\ \Pi\\ \theta\end{array} \quad \begin{array}{c}\Gamma\,,\overline{B}^{(1)}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle(1)}\;\right]}{\theta/\bot}}{\theta/\bot}{\scriptstyle(2)}$$

$$\exists\vee\qquad \left[\;\dfrac{\exists x\varphi \qquad \begin{array}{c}\Xi,\overline{\varphi^x_a}^{(2)}\\ \Theta\\ A\vee B\end{array}}{A\vee B}{\scriptstyle(2)} \qquad \dfrac{A\vee B \quad \begin{array}{c}\Delta,\overline{A}^{(1)}\\ \Pi\\ \theta\end{array} \quad \begin{array}{c}\Gamma\,,\overline{B}^{(1)}\\ \Sigma\\ \bot\end{array}}{\theta}{\scriptstyle(1)}\;\right] \;=\; \dfrac{\exists x\varphi \qquad \dfrac{\left[\;\begin{array}{c}\Xi,\overline{\varphi^x_{a^*}}^{(2)}\\ \Theta\\ A\vee B\end{array} \quad \dfrac{A\vee B \quad \begin{array}{c}\Delta,\overline{A}^{(1)}\\ \Pi\\ \theta\end{array} \quad \begin{array}{c}\Gamma\,,\overline{B}^{(1)}\\ \Sigma\\ \bot\end{array}}{\theta}{\scriptstyle(1)}\;\right]}{\theta/\bot}}{\theta/\bot}{\scriptstyle(2)}$$

$$\exists\vee\qquad \left[\;\dfrac{\exists x\varphi \qquad \begin{array}{c}\Xi,\overline{\varphi^x_a}^{(2)}\\ \Theta\\ A\vee B\end{array}}{A\vee B}{\scriptstyle(2)} \qquad \dfrac{A\vee B \quad \begin{array}{c}\Delta,\overline{A}^{(1)}\\ \Pi\\ \bot\end{array} \quad \begin{array}{c}\Gamma\,,\overline{B}^{(1)}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle(1)}\;\right] \;=\; \dfrac{\exists x\varphi \qquad \dfrac{\left[\;\begin{array}{c}\Xi,\overline{\varphi^x_{a^*}}^{(2)}\\ \Theta\\ A\vee B\end{array} \quad \dfrac{A\vee B \quad \begin{array}{c}\Delta,\overline{A}^{(1)}\\ \Pi\\ \bot\end{array} \quad \begin{array}{c}\Gamma\,,\overline{B}^{(1)}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle(1)}\;\right]}{\theta/\bot}}{\theta/\bot}{\scriptstyle(2)}$$

$$\forall\vee\qquad \left[\;\dfrac{\forall x\varphi \qquad \begin{array}{c}\Xi,\overline{\varphi^x_{t_1}}^{(2)},\ldots,\overline{\varphi^x_{t_n}}^{(2)}\\ \Theta\\ A\vee B\end{array}}{A\vee B}{\scriptstyle(2)} \qquad \dfrac{A\vee B \quad \begin{array}{c}\Delta,\overline{A}^{(1)}\\ \Pi\\ \theta\end{array} \quad \begin{array}{c}\Gamma\,,\overline{B}^{(1)}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle(1)}\;\right] \;=\; \dfrac{\forall x\varphi \qquad \dfrac{\left[\;\begin{array}{c}\Xi,\overline{\varphi^x_{t_1}}^{(2)},\ldots,\overline{\varphi^x_{t_n}}^{(2)}\\ \Theta\\ A\vee B\end{array} \quad \dfrac{A\vee B \quad \begin{array}{c}\Delta,\overline{A}^{(1)}\\ \Pi\\ \theta\end{array} \quad \begin{array}{c}\Gamma\,,\overline{B}^{(1)}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle(1)}\;\right]}{\theta/\bot}}{\theta/\bot}{\scriptstyle(2)}$$

$\forall\vee$

$$\left[\;\dfrac{\forall x\varphi \qquad \begin{array}{c}\Xi,\overline{\varphi^x_{t_1}}^{(2)},\ldots,\overline{\varphi^x_{t_n}}^{(2)}\\ \Theta\\ A\vee B\end{array}}{A\vee B}{\scriptstyle(2)} \qquad \dfrac{A\vee B \quad \begin{array}{c}\Delta,\overline{A}^{(1)}\\ \Pi\\ \theta\end{array} \quad \begin{array}{c}\Gamma,\overline{B}^{(1)}\\ \Sigma\\ \bot\end{array}}{\theta}{\scriptstyle(1)}\;\right] = \dfrac{\forall x\varphi \qquad \dfrac{\left[\;\dfrac{\begin{array}{c}\Xi,\overline{\varphi^x_{t_1}}^{(2)},\ldots,\overline{\varphi^x_{t_n}}^{(2)}\\ \Theta\\ A\vee B\end{array} \quad \begin{array}{c}\Delta,\overline{A}^{(1)}\\ \Pi\\ \theta\end{array} \quad \begin{array}{c}\Gamma,\overline{B}^{(1)}\\ \Sigma\\ \bot\end{array}}{\theta}{\scriptstyle(1)}\;\right]}{\theta/\bot}}{\theta/\bot}{\scriptstyle(2)}$$

$\forall\vee$

$$\left[\;\dfrac{\forall x\varphi \qquad \begin{array}{c}\Xi,\overline{\varphi^x_{t_1}}^{(2)},\ldots,\overline{\varphi^x_{t_n}}^{(2)}\\ \Theta\\ A\vee B\end{array}}{A\vee B}{\scriptstyle(2)} \qquad \dfrac{A\vee B \quad \begin{array}{c}\Delta,\overline{A}^{(1)}\\ \Pi\\ \bot\end{array} \quad \begin{array}{c}\Gamma,\overline{B}^{(1)}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle(1)}\;\right] = \dfrac{\forall x\varphi \qquad \dfrac{\left[\;\dfrac{\begin{array}{c}\Xi,\overline{\varphi^x_{t_1}}^{(2)},\ldots,\overline{\varphi^x_{t_n}}^{(2)}\\ \Theta\\ A\vee B\end{array} \quad \begin{array}{c}\Delta,\overline{A}^{(1)}\\ \Pi\\ \bot\end{array} \quad \begin{array}{c}\Gamma,\overline{B}^{(1)}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle(1)}\;\right]}{\theta/\bot}}{\theta/\bot}{\scriptstyle(2)}$$

$\wedge\rightarrow$

$$\left[\;\dfrac{\varphi\wedge\psi \qquad \begin{array}{c}\underbrace{\Xi\ \ \overline{\varphi}^{(2)}\ \ \overline{\psi}^{(2)}}_{\displaystyle\Theta}\\ A\to B\end{array}}{A\to B}{\scriptstyle(2)} \qquad \dfrac{A\to B \quad \begin{array}{c}\Delta\\ \Pi\\ A\end{array} \quad \begin{array}{c}\Gamma,\overline{B}^{(1)}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle(1)}\;\right] = \dfrac{\varphi\wedge\psi \qquad \dfrac{\left[\;\begin{array}{c}\underbrace{\Xi\ \ \overline{\varphi}^{(2)}\ \ \overline{\psi}^{(2)}}_{\displaystyle\Theta}\\ A\to B\end{array} \quad \dfrac{A\to B \quad \begin{array}{c}\Delta\\ \Pi\\ A\end{array} \quad \begin{array}{c}\Gamma,\overline{B}^{(1)}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle(1)}\;\right]}{\theta/\bot}}{\theta/\bot}{\scriptstyle(2)}$$

$\vee\rightarrow$

$$\left[\;\dfrac{\varphi\vee\psi \quad \begin{array}{c}\overline{\varphi}^{(2)}\\ \Theta_1\\ A\to B\end{array} \quad \begin{array}{c}\overline{\psi}^{(2)}\\ \Theta_2\\ A\to B\end{array}}{A\to B}{\scriptstyle(2)} \qquad \dfrac{A\to B \quad \begin{array}{c}\Delta\\ \Pi\\ A\end{array} \quad \begin{array}{c}\Gamma,\overline{B}^{(1)}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle(1)}\;\right]$$

$$= \quad \dfrac{\varphi\vee\psi \qquad \dfrac{\left[\begin{array}{c}\overline{\varphi}^{(2)}\\ \Theta_1\\ A\to B\end{array} \quad \dfrac{A\to B \quad \begin{array}{c}\Delta\\ \Pi\\ A\end{array} \quad \begin{array}{c}\Gamma,\overline{B}^{(1)}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle(1)}\right]}{\theta/\bot} \qquad \dfrac{\left[\begin{array}{c}\overline{\psi}^{(2)}\\ \Theta_2\\ A\to B\end{array} \quad \dfrac{A\to B \quad \begin{array}{c}\Delta\\ \Pi\\ A\end{array} \quad \begin{array}{c}\Gamma,\overline{B}^{(1)}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle(1)}\right]}{\theta/\bot}}{\theta/\bot}{\scriptstyle(2)}$$

$$\to\to \quad \left[\dfrac{\varphi\to\psi \quad \begin{array}{c}\Xi_1\\ \Theta_1\\ \varphi\end{array} \quad \begin{array}{c}\Xi_2,\overline{\psi}^{(2)}\\ \Theta_2\\ A\to B\end{array}}{A\to B}{\scriptstyle(2)} \quad \dfrac{A\to B \quad \begin{array}{c}\Delta\\ \Pi\\ A\end{array} \quad \begin{array}{c}\Gamma,\overline{B}^{(1)}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle(1)}\right]$$

$$= \quad \dfrac{\varphi\to\psi \quad \begin{array}{c}\Xi_1\\ \Theta_1\\ \varphi\end{array} \quad \dfrac{\left[\begin{array}{c}\Xi_2,\overline{\psi}^{(2)}\\ \Theta_2\\ A\to B\end{array} \quad \dfrac{A\to B \quad \begin{array}{c}\Delta\\ \Pi\\ A\end{array} \quad \begin{array}{c}\Gamma,\overline{B}^{(1)}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle(1)}\right]}{\theta/\bot}}{\theta/\bot}{\scriptstyle(2)}$$

$$\exists\to \quad \left[\dfrac{\exists x\varphi \quad \begin{array}{c}\Xi,\overline{\varphi^x_a}^{(2)}\\ \Theta\\ A\to B\end{array}}{A\to B}{\scriptstyle(2)} \quad \dfrac{A\to B \quad \begin{array}{c}\Delta\\ \Pi\\ A\end{array} \quad \begin{array}{c}\Gamma,\overline{B}^{(1)}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle(1)}\right] = \quad \dfrac{\exists x\varphi \quad \dfrac{\left[\begin{array}{c}\Xi,\overline{\varphi^x_{a^*}}^{(2)}\\ \Theta\\ A\to B\end{array} \quad \dfrac{A\to B \quad \begin{array}{c}\Delta\\ \Pi\\ A\end{array} \quad \begin{array}{c}\Gamma,\overline{B}^{(1)}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle(1)}\right]}{\theta/\bot}}{\theta/\bot}{\scriptstyle(2)}$$

$\forall\to$

$$\left[\;\dfrac{\forall x\varphi \qquad \begin{array}{c}\Xi,\overline{\varphi^x_{t_1}}^{(2)},\ldots,\overline{\varphi^x_{t_n}}^{(2)}\\ \Theta\\ A\to B\end{array}}{A\to B}\,(2) \qquad \dfrac{A\to B \quad \begin{array}{c}\Delta\\ \Pi\\ A\end{array} \quad \begin{array}{c}\Gamma\,,\overline{B}^{(1)}\\ \Sigma\\ \theta\end{array}}{\theta}\,(1)\;\right] \;=\; \dfrac{\forall x\varphi \qquad \dfrac{\left[\;\begin{array}{c}\Xi,\overline{\varphi^x_{t_1}}^{(2)},\ldots,\overline{\varphi^x_{t_n}}^{(2)}\\ \Theta\\ A\to B\end{array} \quad \dfrac{A\to B \quad \begin{array}{c}\Delta\\ \Pi\\ A\end{array} \quad \begin{array}{c}\Gamma\,,\overline{B}^{(1)}\\ \Sigma\\ \theta\end{array}}{\theta}\,(1)\;\right]}{\theta/\bot}}{\theta/\bot}\,(2)$$

$\wedge\exists$

$$\left[\;\dfrac{\varphi\wedge\psi \qquad \begin{array}{c}\underbrace{\Xi\quad\overline{\varphi}^{(2)}\quad\overline{\psi}^{(2)}}_{\Theta}\\ \exists yA\end{array}}{\exists yA}\,(2) \qquad \dfrac{\exists yA \quad \begin{array}{c}\Gamma,\overline{A^y_b}^{(1)}\\ \Sigma\\ \theta\end{array}}{\theta}\,(1)\;\right] \;=\; \dfrac{\varphi\wedge\psi \qquad \dfrac{\left[\;\begin{array}{c}\underbrace{\Xi\quad\overline{\varphi}^{(2)}\quad\overline{\psi}^{(2)}}_{\Theta}\\ \exists yA\end{array} \quad \dfrac{\exists yA \quad \begin{array}{c}\Gamma,\overline{A^y_b}^{(1)}\\ \Sigma\\ \theta\end{array}}{\theta}\,(1)\;\right]}{\theta/\bot}}{\theta/\bot}\,(2)$$

$\vee\exists$

$$\left[\;\dfrac{\varphi\vee\psi \quad \begin{array}{c}\overline{\varphi}^{(2)}\\ \Theta_1\\ \exists yA\end{array} \quad \begin{array}{c}\overline{\psi}^{(2)}\\ \Theta_2\\ \exists yA\end{array}}{\exists yA}\,(2) \qquad \dfrac{\exists yA \quad \begin{array}{c}\Gamma,\overline{A^y_b}^{(1)}\\ \Sigma\\ \theta\end{array}}{\theta}\,(1)\;\right]$$

$$=\; \dfrac{\varphi\vee\psi \qquad \dfrac{\left[\;\begin{array}{c}\overline{\varphi}^{(2)}\\ \Theta_1\\ \exists yA\end{array} \quad \dfrac{\exists yA \quad \begin{array}{c}\Gamma,\overline{A^y_b}^{(1)}\\ \Sigma\\ \theta\end{array}}{\theta}\,(1)\;\right]}{\theta/\bot} \qquad \dfrac{\left[\;\begin{array}{c}\overline{\psi}^{(2)}\\ \Theta_2\\ \exists yA\end{array} \quad \dfrac{\exists yA \quad \begin{array}{c}\Gamma,\overline{A^y_b}^{(1)}\\ \Sigma\\ \theta\end{array}}{\theta}\,(1)\;\right]}{\theta/\bot}}{\theta/\bot}\,(2)$$

$\to\exists$

$$\left[\;\dfrac{\varphi\to\psi \qquad \begin{array}{c}\Xi_1\\ \Theta_1\\ \varphi\end{array} \qquad \begin{array}{c}\Xi_2, \overline{\psi}^{(2)}\\ \Theta_2\\ \exists yA\end{array}}{\exists yA}{\scriptstyle(2)} \qquad \dfrac{\exists yA \qquad \begin{array}{c}\Gamma, \overline{A^y_b}^{(1)}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle(1)}\;\right] = \dfrac{\varphi\to\psi \qquad \begin{array}{c}\Xi_1\\ \Theta_1\\ \varphi\end{array} \qquad \dfrac{\left[\begin{array}{cc}\begin{array}{c}\Xi_2, \overline{\psi}^{(2)}\\ \Theta_2\\ \exists yA\end{array} & \dfrac{\exists yA \qquad \begin{array}{c}\Gamma, \overline{A^y_b}^{(1)}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle(1)}\end{array}\right]}{\theta/\bot}}{\theta/\bot}{\scriptstyle(2)}$$

$\exists\exists$

$$\left[\;\dfrac{\exists x\varphi \qquad \begin{array}{c}\Xi, \overline{\varphi^x_a}^{(2)}\\ \Theta\\ \exists yA\end{array}}{\exists yA}{\scriptstyle(2)} \qquad \dfrac{\exists yA \qquad \begin{array}{c}\Gamma, \overline{A^y_b}^{(1)}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle(1)}\;\right] = \dfrac{\exists x\varphi \qquad \dfrac{\left[\begin{array}{cc}\begin{array}{c}\Xi, \overline{\varphi^x_{a^*}}^{(2)}\\ \Theta\\ \exists yA\end{array} & \dfrac{\exists yA \qquad \begin{array}{c}\Gamma, \overline{A^y_b}^{(1)}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle(1)}\end{array}\right]}{\theta/\bot}}{\theta/\bot}{\scriptstyle(2)}$$

$\forall\exists$

$$\left[\;\dfrac{\forall x\varphi \qquad \begin{array}{c}\Xi, \overline{\varphi^x_{t_1}}^{(2)}, \ldots, \overline{\varphi^x_{t_n}}^{(2)}\\ \Theta\\ \exists yA\end{array}}{\exists yA}{\scriptstyle(2)} \qquad \dfrac{\exists yA \qquad \begin{array}{c}\Gamma, \overline{A^y_b}^{(1)}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle(1)}\;\right] = \dfrac{\forall x\varphi \qquad \dfrac{\left[\begin{array}{cc}\begin{array}{c}\Xi, \overline{\varphi^x_{t_1}}^{(2)}, \ldots, \overline{\varphi^x_{t_n}}^{(2)}\\ \Theta\\ \exists yA\end{array} & \dfrac{\exists yA \qquad \begin{array}{c}\Gamma, \overline{A^y_b}^{(1)}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle(1)}\end{array}\right]}{\theta/\bot}}{\theta/\bot}{\scriptstyle(2)}$$

$\wedge\forall$

$$\left[\;\dfrac{\varphi\wedge\psi \qquad \begin{array}{c}\underbrace{\Xi \quad \overline{\varphi}^{(2)} \quad \overline{\psi}^{(2)}}\\ \Theta\\ \forall yA\end{array}}{\forall yA}{\scriptstyle(2)} \qquad \dfrac{\forall yA \qquad \begin{array}{c}\Gamma, \overline{A^y_{t_1}}^{(1)}, \ldots, \overline{A^y_{t_n}}^{(1)}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle(1)}\;\right] = \dfrac{\varphi\wedge\psi \qquad \dfrac{\left[\begin{array}{cc}\begin{array}{c}\underbrace{\Xi \quad \overline{\varphi}^{(2)} \quad \overline{\psi}^{(2)}}\\ \Theta\\ \forall yA\end{array} & \dfrac{\forall yA \qquad \begin{array}{c}\Gamma, \overline{A^y_{t_1}}^{(1)}, \ldots, \overline{A^y_{t_n}}^{(1)}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle(1)}\end{array}\right]}{\theta/\bot}}{\theta/\bot}{\scriptstyle(2)}$$

$$\vee\forall \qquad \left[\; \dfrac{\varphi\vee\psi \quad \begin{array}{c}\overline{\varphi}^{(2)}\\ \Theta_1\\ \forall yA\end{array} \quad \begin{array}{c}\overline{\psi}^{(2)}\\ \Theta_2\\ \forall yA\end{array}}{\forall yA}{\scriptstyle (2)} \qquad \dfrac{\forall yA \quad \begin{array}{c}\Gamma, \overline{A^y_{t_1}}^{(1)},\dots,\overline{A^y_{t_n}}^{(1)}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle (1)}\;\right]$$

$$= \dfrac{\varphi\vee\psi \quad \dfrac{\left[\begin{array}{c}\overline{\varphi}^{(2)}\\ \Theta_1\\ \forall yA\end{array} \quad \dfrac{\forall yA \quad \begin{array}{c}\Gamma, \overline{A^y_{t_1}}^{(1)},\dots,\overline{A^y_{t_n}}^{(1)}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle (1)}\right]}{\theta/\bot} \quad \dfrac{\left[\begin{array}{c}\overline{\psi}^{(2)}\\ \Theta_2\\ \forall yA\end{array} \quad \dfrac{\forall yA \quad \begin{array}{c}\Gamma, \overline{A^y_{t_1}}^{(1)},\dots,\overline{A^y_{t_n}}^{(1)}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle (1)}\right]}{\theta/\bot}}{\theta/\bot}{\scriptstyle (2)}$$

$$\to\forall \qquad \left[\; \dfrac{\varphi\to\psi \quad \begin{array}{c}\Xi_1\\ \Theta_1\\ \varphi\end{array} \quad \begin{array}{c}\Xi_2,\overline{\psi}^{(2)}\\ \Theta_2\\ \forall yA\end{array}}{\forall yA}{\scriptstyle (2)} \qquad \dfrac{\forall yA \quad \begin{array}{c}\Gamma, \overline{A^y_{t_1}}^{(1)},\dots,\overline{A^y_{t_n}}^{(1)}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle (1)}\;\right] = \dfrac{\varphi\to\psi \quad \begin{array}{c}\Xi_1\\ \Theta_1\\ \varphi\end{array} \quad \dfrac{\left[\begin{array}{c}\Xi_2,\overline{\psi}^{(2)}\\ \Theta_2\\ \forall yA\end{array} \quad \dfrac{\forall yA \quad \begin{array}{c}\Gamma, \overline{A^y_{t_1}}^{(1)},\dots,\overline{A^y_{t_n}}^{(1)}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle (1)}\right]}{\theta/\bot}}{\theta/\bot}{\scriptstyle (2)}$$

$$\exists\forall \qquad \left[\; \dfrac{\exists x\varphi \quad \begin{array}{c}\Xi,\overline{\varphi^x_a}^{(2)}\\ \Theta\\ \forall yA\end{array}}{\forall yA}{\scriptstyle (2)} \qquad \dfrac{\forall yA \quad \begin{array}{c}\Gamma, \overline{A^y_{t_1}}^{(1)},\dots,\overline{A^y_{t_n}}^{(1)}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle (1)}\;\right] = \dfrac{\exists x\varphi \quad \dfrac{\left[\begin{array}{c}\Xi,\overline{\varphi^x_{a^*}}^{(2)}\\ \Theta\\ \forall yA\end{array} \quad \dfrac{\forall yA \quad \begin{array}{c}\Gamma, \overline{A^y_{t_1}}^{(1)},\dots,\overline{A^y_{t_n}}^{(1)}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle (1)}\right]}{\theta/\bot}}{\theta/\bot}{\scriptstyle (2)}$$

$$\forall\forall\qquad \left[\begin{array}{cc} \dfrac{\forall x\varphi \qquad \begin{array}{c}\Xi, \overline{\varphi^x_{t_1}}^{(2)},\ldots,\overline{\varphi^x_{t_n}}^{(2)}\\ \Theta \\ \forall yA\end{array}}{\forall yA}{\scriptstyle(2)} & \dfrac{\forall yA \qquad \begin{array}{c}\Gamma, \overline{A^y_{t_1}}^{(1)},\ldots,\overline{A^y_{t_n}}^{(1)}\\ \Sigma \\ \theta\end{array}}{\theta}{\scriptstyle(1)}\end{array}\right] = \dfrac{\forall x\varphi \qquad \dfrac{\left[\dfrac{\begin{array}{cc}\begin{array}{c}\Xi, \overline{\varphi^x_{t_1}}^{(2)},\ldots,\overline{\varphi^x_{t_n}}^{(2)}\\ \Theta \\ \forall yA\end{array} & \begin{array}{c}\Gamma, \overline{A^y_{t_1}}^{(1)},\ldots,\overline{A^y_{t_n}}^{(1)}\\ \Sigma \\ \theta\end{array}\end{array}}{\theta}{\scriptstyle(1)}\right]}{\theta/\bot}}{\theta/\bot}{\scriptstyle(2)}$$

That completes our list of permutation conversions. We turn finally to the reductions $[\Pi\ \Sigma]$, which are called for when the cut-sentence A is the conclusion of an introduction (at the last step in Π) and the major premise of the corresponding elimination (at the last step in Σ).

*Assume that the occurrence of A as the MPE of the terminal step in Σ is the **only** assumption-occurrence of A in Σ.* Then the following transformations apply.

$$\neg_I\neg\qquad \left[\begin{array}{cc} \dfrac{\begin{array}{c}{}^{(i)}\overline{A}, \Xi\\ \Theta\\ \bot\end{array}}{\neg A}{\scriptstyle(i)} & \dfrac{\neg A \qquad \begin{array}{c}\Delta\\ \Pi\\ A\end{array}}{\bot}{\scriptstyle(1)}\end{array}\right] = \left[\begin{array}{cc}\Delta & A, \Xi\\ \Pi & \Theta\\ A & \bot\end{array}\right]$$

$$\wedge_I\wedge\qquad \left[\begin{array}{cc} \dfrac{\begin{array}{cc}\Xi_1 & \Xi_2\\ \Theta_1 & \Theta_2\\ A & B\end{array}}{A\wedge B} & \dfrac{A\wedge B \qquad \begin{array}{c}\Gamma, \overline{A}^{(1)}, \overline{B}^{(1)}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle(1)}\end{array}\right] = \left[\begin{array}{cc}\begin{array}{c}\Xi_1\\ \Theta_1\\ A\end{array} & \left[\begin{array}{cc}\Xi_2 & \Gamma, A, B\\ \Theta_2 & \Sigma\\ B & \theta\end{array}\right]\end{array}\right]$$

$$\vee_I\vee\qquad \left[\begin{array}{cc} \dfrac{\begin{array}{c}\Xi_i\\ \Theta_i\\ A_i\end{array}}{A_1\vee A_2} & \dfrac{A_1\vee A_2 \qquad \begin{array}{c}{}^{(j)}\overline{A_1}, \Gamma_1\\ \Sigma_1\\ \theta\end{array} \qquad \begin{array}{c}{}^{(j)}\overline{A_2}, \Gamma_2\\ \Sigma_2\\ \theta\end{array}}{\theta}{\scriptstyle(j)}\end{array}\right] = \left[\begin{array}{cc}\Xi_i & A_i, \Gamma_i\\ \Theta_i & \Sigma_i\\ A_i & \theta\end{array}\right] \qquad (i = 1, 2)$$

$$\vee_I\vee \qquad \left[\begin{array}{cc} \begin{array}{c}\Xi_1\\ \Theta_1\\ \dfrac{A_1}{A_1\vee A_2}\end{array} & \dfrac{A_1\vee A_2 \quad \begin{array}{c}{}^{(j)}\overline{A_1,\Gamma_1}\\ \Sigma_1\\ \theta\end{array} \quad \begin{array}{c}{}^{(j)}\overline{A_2,\Gamma_2}\\ \Sigma_2\\ \bot\end{array}}{\theta}{\scriptstyle (j)}\end{array}\right] = \left[\begin{array}{cc}\Xi_1 & A_1,\Gamma_1\\ \Theta_1 & \Sigma_1\\ A_1 & \theta\end{array}\right]$$

$$\vee_I\vee \qquad \left[\begin{array}{cc} \begin{array}{c}\Xi_2\\ \Theta_2\\ \dfrac{A_2}{A_1\vee A_2}\end{array} & \dfrac{A_1\vee A_2 \quad \begin{array}{c}{}^{(j)}\overline{A_1,\Gamma_1}\\ \Sigma_1\\ \theta\end{array} \quad \begin{array}{c}{}^{(j)}\overline{A_2,\Gamma_2}\\ \Sigma_2\\ \bot\end{array}}{\theta}{\scriptstyle (j)}\end{array}\right] = \left[\begin{array}{cc}\Xi_2 & A_2,\Gamma_2\\ \Theta_2 & \Sigma_2\\ A_2 & \bot\end{array}\right]$$

$$\vee_I\vee \qquad \left[\begin{array}{cc} \begin{array}{c}\Xi_1\\ \Theta_1\\ \dfrac{A_1}{A_1\vee A_2}\end{array} & \dfrac{A_1\vee A_2 \quad \begin{array}{c}{}^{(j)}\overline{A_1,\Gamma_1}\\ \Sigma_1\\ \bot\end{array} \quad \begin{array}{c}{}^{(j)}\overline{A_2,\Gamma_2}\\ \Sigma_2\\ \theta\end{array}}{\theta}{\scriptstyle (j)}\end{array}\right] = \left[\begin{array}{cc}\Xi_1 & A_1,\Gamma_1\\ \Theta_1 & \Sigma_1\\ A_1 & \bot\end{array}\right]$$

$$\vee_I\vee \qquad \left[\begin{array}{cc} \begin{array}{c}\Xi_2\\ \Theta_2\\ \dfrac{A_2}{A_1\vee A_2}\end{array} & \dfrac{A_1\vee A_2 \quad \begin{array}{c}{}^{(j)}\overline{A_1,\Gamma_1}\\ \Sigma_1\\ \bot\end{array} \quad \begin{array}{c}{}^{(j)}\overline{A_2,\Gamma_2}\\ \Sigma_2\\ \theta\end{array}}{\theta}{\scriptstyle (j)}\end{array}\right] = \left[\begin{array}{cc}\Xi_2 & A_2,\Gamma_2\\ \Theta_2 & \Sigma_2\\ A_2 & \theta\end{array}\right]$$

$$\to_I\to \qquad \left[\begin{array}{cc} \begin{array}{c}{}^{(i)}\overline{A}\,,\,\Xi\\ \Theta\\ \dfrac{\bot}{A\to B}{\scriptstyle (i)}\end{array} & \dfrac{A\to B \quad \begin{array}{c}\Delta\\ \Pi\\ A\end{array} \quad \begin{array}{c}{}^{(j)}\overline{B}\,,\,\Gamma\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle (j)}\end{array}\right] = \left[\begin{array}{cc}\Delta & A\,,\,\Xi\\ \Pi & \Theta\\ A & \bot\end{array}\right]$$

$$\to_I\to \quad \left[\begin{array}{cc} \begin{array}[b]{c} {}^{(i)}\overline{A}\,,\ \Xi \\ \Theta \\ \dfrac{B}{A\to B}{\scriptstyle(i)} \end{array} & \dfrac{A\to B \quad \begin{array}[b]{c}\Delta\\ \Pi\\ A\end{array} \quad \begin{array}[b]{c}{}^{(j)}\overline{B}\,,\ \Gamma\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle(j)} \end{array}\right] = \left[\begin{array}{c}\Delta\\ \Pi\\ A\end{array} \left[\begin{array}{cc} A\,,\ \Xi & B\,,\ \Gamma \\ \Theta & \Sigma \\ B & \theta \end{array}\right]\right]$$

$$\exists_I\exists \quad \left[\begin{array}{cc} \begin{array}[b]{c}\Xi\\ \Theta\\ \dfrac{A^x_t}{\exists xA}\end{array} & \dfrac{\exists xA \quad \begin{array}[b]{c}{}^{(i)}\overline{A^x_a}\,,\ \Gamma\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle(i)}\end{array}\right] = \left[\begin{array}{cc}\Xi & A^x_t\,,\ \Gamma\\ \Theta & \Sigma^a_t\\ A^x_t & \theta\end{array}\right]$$

$$\forall_I\forall \quad \left[\begin{array}{cc} \begin{array}[b]{c}\Xi\\ \Theta\\ \dfrac{A}{\forall xA^a_x}\end{array} & \dfrac{\forall xA^a_x \quad \begin{array}[b]{c}\underbrace{{}^{(i)}\overline{A^a_{t_1}}\ ,\ldots,\ {}^{(i)}\overline{A^a_{t_n}}\,,\ \Gamma}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle(i)}\end{array}\right] = \left[\begin{array}{c}\Xi\\ \Theta^a_{t_1}\\ A^a_{t_1}\end{array}\ \ldots \left[\begin{array}{cc}\Xi & \underbrace{A^a_{t_1},\ldots,A^a_{t_n}\,,\ \Gamma}\\ \Theta^a_{t_n} & \Sigma\\ A^a_{t_n} & \theta\end{array}\right]\ldots\right]$$

Now assume that the occurrence of A as the MPE of the terminal step in Σ is ***not*** *the only undischarged MPE-occurrence of A in Σ.* Perform the following operation:

> Consider the leftmost among the highest of the non-terminal undischarged MPE-occurrences of A in Σ. Denote by Σ^A the subproof of Σ whose final step is the elimination in question. Replace Σ^A in Σ by $[\Pi\,\Sigma^A]$. Call the result $\Sigma^{\Sigma^A}_{[\Pi\,\Sigma^A]}$. Now determine $[\Pi\ \Sigma^{\Sigma^A}_{[\Pi\,\Sigma^A]}]$.

Repeat this operation until the only undischarged MPE-occurrence of A within the resulting proof is the one that was terminal in Σ, so that one of the immediately foregoing transformations will apply.

With our inductive definition of $[\Pi\,\Sigma]$, it is clear that every transformation or operation either reduces the complexity of the cut sentence in hand, or reduces the number of its undischarged assumption-occurrences within Σ.

Theorem 1 has now been proved.

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