Exercise 9 in *Natural Logic*, on p.186.

Prove, or construct a counterexample to, each of the following arguments:

$$\frac{\forall x \exists y Rxy \qquad \exists y \forall z Ryz}{\forall x \forall z Rxz}$$

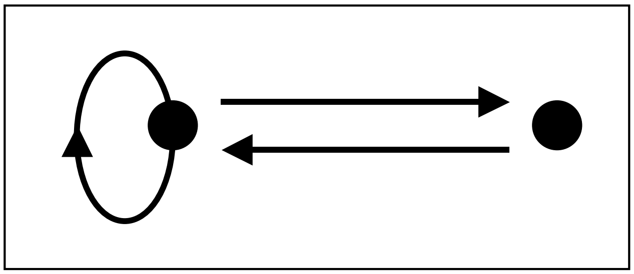
$$\frac{\forall x \neg Rxx \qquad \forall x \forall y (Rxy \rightarrow \neg Ryx)}{\forall x \forall y \forall z ((Rxy \wedge Ryz) \rightarrow Rxz)}$$

$$\frac{\forall x \exists y Rxy \qquad \forall x \exists y Sxy}{\forall x \exists \mathrm{y} (Rxy \wedge Sxy)}$$

$$\frac{\forall x \forall y \forall z ((Rxy \wedge Ryz) \rightarrow Rxz) \qquad \forall x \neg Rxx}{\forall x \forall y (Rxy \rightarrow \neg Ryx)}$$

$$\frac{\forall x \exists y Rxy \qquad \exists y \forall z Ryz}{\forall x \forall z Rxz}$$

This argument is invalid. A *finite* counterexample is as follows:



A simple *infinite* counterexample can be obtained by interpreting Rxy as $x \leq y$ on all positive integers (including 0). We have

$\forall x \exists y Rxy$ is true,

since every integer is less than or equal to itself;

$\exists y \forall z Ryz$ is true,

since 0 is less than or equal to each positive integer;

but

$\forall x \forall z Rxz$ is false,

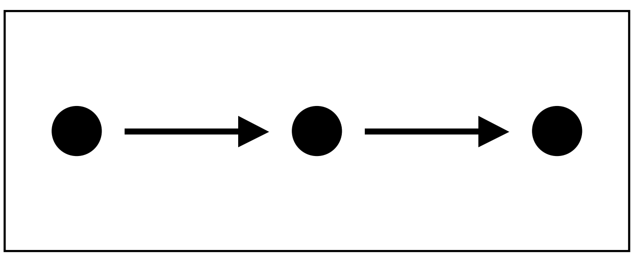
since (for example) it is false that $1 \leq 0$.

$$\frac{\forall x \neg Rxx \qquad \forall x \forall y (Rxy \rightarrow \neg Ryx)}{\forall x \forall y \forall z ((Rxy \wedge Ryz) \rightarrow Rxz)}$$

The first premise follows logically from the second premise. So the problem reduces to that of assessing the validity of the argument

$$\frac{\forall x \forall y (Rxy \rightarrow \neg Ryx)}{\forall x \forall y \forall z ((Rxy \wedge Ryz) \rightarrow Rxz)}$$

The argument is invalid. A simple counterexample is

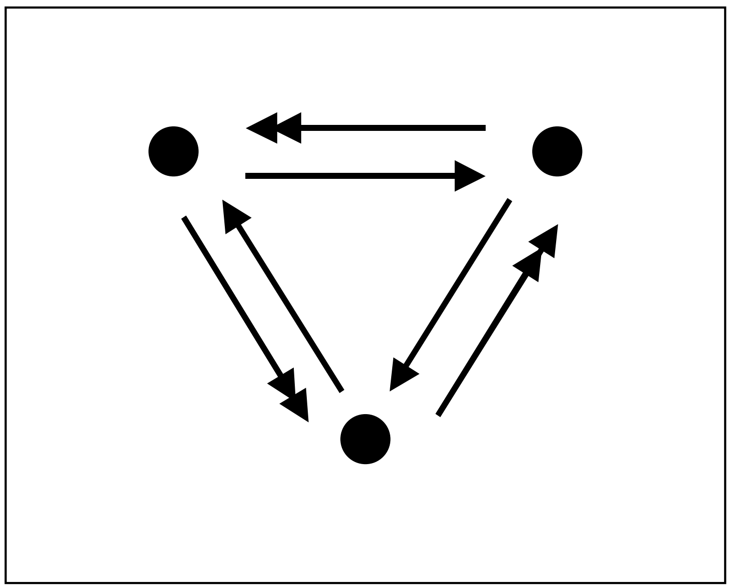


The premise $\forall x \forall y (Rxy \rightarrow \neg Ryx)$ is true, since the arrows are one-way.

The conclusion is falsified by taking the dots from left to right for x, y, z respectively.

$$\frac{\forall x \exists y Rxy \qquad \forall x \exists y Sxy}{\forall x \exists \mathrm{y} (Rxy \wedge Sxy)}$$

The argument is invalid. A simple counterexample is



with *R* and *S* interpreted by single- and double-headed arrows respectively.

$$\frac{\forall x\forall y\forall z((Rxy\wedge Ryz)\to Rxz) \qquad \forall x\neg Rxx}{\forall x\forall y(Rxy\to\neg Ryx)}$$

The argument is valid. Here is a proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall x\forall y\forall z((Rxy\wedge Ryz)\to Rxz)}{\forall y\forall z((Ray\wedge Ryz)\to Raz)}}{\forall z((Rab\wedge Rbz)\to Raz)}}{(Rab\wedge Rba)\to Raa} \qquad \dfrac{\dfrac{}{Rab}{\scriptstyle(2)} \quad \dfrac{}{Rba}{\scriptstyle(1)}}{Rab\wedge Rba}}{Raa} \qquad \dfrac{\forall x\neg Rxx}{\neg Raa}}{\bot}{\scriptstyle(1)}}{\neg Rba}{\scriptstyle(2)}}{Rab\to\neg Rba}}{\forall y(Ray\to\neg Rya)}}{\forall x\forall y(Rxy\to\neg Ryx)}$$