PHIL650: INTRODUCTION TO SYMBOLIC LOGIC.
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Definition. $x \subseteq y \equiv_{df} \forall z(z \in x \rightarrow z \in y)$.

Definition. $\{x \in X | \psi(x)\} =_{df} \{x | x \in X \wedge \psi(x)\}$.

CANTOR'S THEOREM.
For no set is there a function mapping its members onto all its subsets.
Proof. Let X be any set. Suppose, for *reductio ad absurdum*, that f is a function mapping the members of X onto all the subsets of X. Let

$$D \ =_{df} \ \{x \in X | x \notin f(x)\}.$$

By the axiom of separation, D exists; and we have $D \subseteq X$. Since f is onto, there must be some $d \in X$ such that $D = f(d)$.

Suppose $d \in D$, i.e. $d \in \{x \in X | x \notin f(x)\}$. Then $d \notin f(d)$, i.e. $d \notin D$. Thus $d \notin D$, i.e. $d \notin f(d)$. But now, since $d \in X$, we have $d \in \{x \in X | x \notin f(x)\}$, i.e. $d \in D$. Contradiction. *QED*

So: there is no function mapping the members of X onto the subsets of X. *A fortiori*, there cannot be any 1-1 relation between all the members of X and all the subsets of X—because such a relation would provide a function of the kind shown to be impossible.

Note that this proof does *not* require that one be able to form the power set of X, i.e. the set of all subsets of X. It shows the impossibility, for any set X, of a formula $\varphi(x, y)$ purporting to represent a function from the members of X to all subsets of X, i.e. a formula $\varphi(x, y)$ for which the following conditions would hold:

$$\forall x(x \in X \rightarrow \exists y(y \subseteq X \wedge \varphi(x, y))$$
$$\forall x \forall y \forall z((\varphi(x, y) \wedge \varphi(x, z)) \rightarrow x = z)$$
$$\forall y(y \subseteq X \rightarrow \exists x(x \in X \wedge \varphi(x, y)))$$

Exercise (for the masochist). Construct a completely formal proof, in a sequent calculus or in a system of natural deduction, of the inconsistency of the last three sentences within the (free) logic of sets. You may use the above definitions; the Axiom (Scheme) of Separation; the rules of intuitionistic logic; and the rules for introducing and eliminating the set-term-forming operator. Let the informal proof above be your guide.