Phil650: Introduction to Symbolic Logic.

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Strong Completeness of Classical First-Order Logic.

$\vdash$ is classical deducibility. We deal here with a language whose only logical operators are $\neg, \wedge$ and $\exists$, and whose only extralogical primitives are names and predicates.

Define Δ, φ to be $\Delta \cup \{\varphi\}$.

Let $A =< a_0, a_1, \ldots >$ be a list of names containing all those occurring in any sentence in Γ, φ, and containing infinitely many names not occurring in any sentence in Γ, φ.

Definition (by cases):

if $\Gamma, \varphi \nvdash \bot$ and φ is not an existential sentence, then

$$\Gamma^{\varphi}_{A} =_{df} \Gamma, \varphi;$$

if $\Gamma, \varphi \nvdash \bot$ and φ is an existential sentence ($= \exists x\psi$, say), then

$$\Gamma^{\varphi}_{A} =_{df} \Gamma, \exists x\psi, \psi^{x}_{a}$$

(where a is the first name in the list A not to occur in $\Gamma, \exists x\psi$);

if $\Gamma, \varphi \vdash \bot$, then

$$\Gamma^{\varphi}_{A} =_{df} \Gamma, \neg\varphi.$$

We call Γ^{φ}_{A} the *Lindenbaum expansion* of Γ with respect to φ and relative to the list A (which provides a definite choice of a name whenever it is called for). The Lindenbaum expansion with respect to φ is obtained from Γ by adding φ if it is consistent to do so, along with a fresh witnessing instance if φ is an existential; otherwise, by adding $\neg\varphi$.

Lindenbaum's Lemma. Lindenbaum expansion preserves consistency; that is,

for all Γ, if $\Gamma \nvdash \bot$ then for all φ, $\Gamma^{\varphi}_{A} \nvdash \bot$.

Proof. Suppose first that $\Gamma, \varphi \nvdash \bot$, and φ is an existential sentence $\exists x\psi$. Then we have

$$\Gamma, \exists x\psi \nvdash \bot \ \text{ and } \ \Gamma^{\varphi}_{A} =_{df} \Gamma, \exists x\psi, \psi^{x}_{a}$$

(where a is the first name not to occur in $\Gamma, \exists x\psi$). Suppose for *reductio* that $\Gamma, \exists x\psi, \psi^{x}_{a} \vdash \bot$, and that the proof of this inconsistency is called Π, say. Then the following proof would exist, its final step of $\exists E$ meeting all parametric constraints because of our choice of a:

$$\dfrac{\exists x\psi \qquad \begin{array}{c}\underbrace{\Gamma,\ \ \dfrac{\overline{\psi^{x}_{a}}^{(1)}}{\exists x\psi},\ \ \overline{\psi^{x}_{a}}^{(1)}}\\ \Pi \\ \bot\end{array}}{\bot}{\scriptstyle(1)}$$

So we would have $\Gamma, \exists x\psi \vdash \bot$, contrary to assumption. Hence $\Gamma, \exists x\psi, \psi^{x}_{a} \nvdash \bot$; that is, $\Gamma^{\varphi}_{A} \nvdash \bot$.

Now suppose that $\Gamma, \varphi \nvdash \bot$, and φ is not an existential sentence. Then clearly $\Gamma^{\varphi}_{A} \nvdash \bot$.

Finally, suppose that $\Gamma, \varphi \vdash \bot$, by virtue of proof Σ, say. Given the proof-schema

$$\begin{array}{c}\underbrace{\Gamma \qquad \dfrac{\begin{array}{c}\underbrace{\Gamma \quad \overline{\varphi}^{(1)}}\\ \Sigma \\ \bot\end{array}}{\neg\varphi}{\scriptstyle(1)}}\\ \Xi \\ \bot\end{array}$$

and the consistency of Γ, it follows that there can be no such proof as Ξ; whence $\Gamma, \neg\varphi \nvdash \bot$. That is, $\Gamma^{\varphi}_{A} \nvdash \bot$.

Definitions.

Γ is *maximal* just in case for every sentence φ, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.

Γ *has witnesses* just in case for every existential sentence $\exists x\psi \in \Gamma$, there is some name a such that $\psi^{x}_{a} \in \Gamma$.

Expansion Lemma. If $\Delta \nvdash \perp$, then Δ can be expanded to a maximal consistent set with witnesses.
Proof. Let $B =< a_0, a_1, \ldots >$ be a list of names containing all those occurring in any sentence in Δ, *and* containing infinitely many names not occurring in any sentence in Δ. Let $\varphi_0, \varphi_1, \ldots$ be a list of all sentences that can be formed by using predicates involved in members of Δ and names drwan from the list B. Define a sequence $\Delta_0, \Delta_1, \ldots$ as follows:

$$\Delta_0 =_{df} \Delta;$$

$$\Delta_{n+1} =_{df} (\Delta_n)_B^{\varphi_n}.$$

Set

$$\overline{\Delta} =_{df} \bigcup_i \Delta_i.$$

We show that $\overline{\Delta}$ is the desired expansion. First, $\overline{\Delta}$ is consistent:

> Δ_0 is consistent, and each expansion step (from Δ_n to Δ_{n+1}) preserves consistency. Hence by induction every stage Δ_i is consistent. Moreover, any proof of $\perp$ from $\overline{\Delta}$ would have all its undischarged assumptions in some Δ_k, contrary to consistency of every stage.

Secondly, it is obvious by the construction of $\overline{\Delta}$ that it is maximal. Thirdly, the method of construction also secures witnesses for every existential claim in $\overline{\Delta}$. That completes the proof.

Recall an earlier lemma to the effect that every maximal consistent set is closed—that is, contains all its provable consequences.

Satisfiability Lemma. Every maximal consistent set with witnesses has a model.
Proof. Let Γ be maximal consistent with witnesses. Define a model M ($= M_\Gamma$) as follows. The individuals in the domain of M are the names occurring in the sentences of Γ. Each name denotes itself. For any n-place primitive predicate P, the n-tuple $<a_1, \ldots, a_n>$ will be in the M-extension of P just in case the sentence $P(a_1, \ldots, a_n)$ is in Γ. That secures the basis step for an inductive proof of the claim that for every sentence φ,

$$\varphi \text{ is true in } M \text{ if and only if } \varphi \in \Gamma$$

—since $P(a_1, \ldots a_n)$ is true in M just in case $< a_1, \ldots, a_n >$ is in the M-extension of P.

Assume now the inductive hypothesis, namely that for every sentence χ simpler than φ,

$$\chi \text{ is true in } M \text{ if and only if } \chi \in \Gamma.$$

We perform the inductive step by cases, according to the dominant operator in φ.

If $\varphi = \neg\psi$, we argue from left to right as follows:

> Suppose $\neg\psi$ is true in M. Then by the truth-rule for negation, ψ is not true in M. Hence by inductive hypothesis $\psi \notin \Gamma$. So, since Γ is maximal, $\neg\psi \in \Gamma$.

The converse is argued as follows:

> Suppose $\neg\psi \in \Gamma$. Since Γ is consistent, the rule of $\neg E$ means we have $\psi \notin \Gamma$. By inductive hypothesis, ψ is not true in M. Thus by the truth-rule for negation, $\neg\psi$ is true in M.

If $\varphi = \psi \wedge \theta$, we argue from left to right as follows:

> Suppose $\psi \wedge \theta$ is true in M. Then by the truth-rule for conjunction, ψ is true in M and θ is true in M. By inductive hypothesis, $\psi \in \Gamma$ and $\theta \in \Gamma$. By $\wedge I$, $\Gamma \vdash \psi \wedge \theta$. So, since Γ is closed, $\psi \wedge \theta \in \Gamma$.

The converse is argued as follows:

> Suppose $\psi \wedge \theta \in \Gamma$. Then by $\wedge E$, $\Gamma \vdash \psi$ and $\Gamma \vdash \theta$. Since Γ is closed, $\psi \in \Gamma$ and $\theta \in \Gamma$. By inductive hypothesis, ψ is true in M and θ is true in M. Hence by the truth-rule for conjunction, $\psi \wedge \theta$ is true in M.

If $\varphi = \exists x\psi$, we argue from left to right as follows:

> Suppose $\exists x\psi$ is true in M. Then for some individual a in the domain of M, $M \models \psi[\overset{x}{\underset{a}{\downarrow}}]$. Since a denotes a in M, we have $M \models \psi^x_a$. By inductive hypothesis, $\psi^x_a \in \Gamma$. By $\exists I$, $\Gamma \vdash \exists x\psi$. Thus since Γ is closed, $\exists x\psi \in \Gamma$.

The converse is argued as follows:

> Suppose $\exists x\psi \in \Gamma$. Since Γ has witnesses, there is some name a such that $\psi^x_a \in \Gamma$. By inductive hypothesis, ψ^x_a is true in M. But a denotes something in the domain of M (namely, a). Hence by the truth-rule for the existential quantifier, $\exists x\psi$ is true in M.

Henkin's Lemma. Every consistent set of sentences has a model.
Proof. Suppose Δ is consistent. Expand Δ to a maximal consistent set $\overline{\Delta}$ with witnesses. This has a model, which is thereby also a model for Δ, since $\Delta \subseteq \overline{\Delta}$.

Strong Completeness Theorem. If $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$.
Proof. Suppose $\Gamma \models \varphi$. Then $\Gamma, \neg\varphi$ has no model. Contraposing Henkin's Lemma, we conclude that $\Gamma, \neg\varphi \vdash \perp$. Hence by classical *reductio*, $\Gamma \vdash \varphi$.

Compactness Theorem. If $\Gamma \models \varphi$ then for some finite $\Delta \subseteq \Gamma$, $\Delta \models \varphi$.
Proof. Suppose $\Gamma \models \varphi$. Then by completeness, $\Gamma \vdash \varphi$. But proofs use only finitely many premisses. Hence for some finite $\Delta \subseteq \Gamma$, we have $\Delta \vdash \varphi$; whence, by soundness of proof, $\Delta \models \varphi$.

Countable Models Theorem. If Δ is countable and consistent, then Δ has a countable model.
Proof. It is clear from the proof of the Expansion Lemma that $\overline{\Delta}$ is countable if Δ is. It is equally clear from the proof of the Satisfiability Lemma that the model constructed for any *countable* maximal consistent set of sentences with witnesses is countable—since its domain consists just of the names involved in those sentences. Hence the result.

Exercise. Work out *exactly* which rules of inference have been invoked in order to establish the foregoing metatheorems for classical first-order logic.