Phil650: Introduction to Symbolic Logic.

Instructor: Neil Tennant. tennant.9@osu.edu Tel. 614-2921591 (UH328)

Strong Completeness of Classical Propositional Logic.

Although our main aim is to establish the completeness of *classical* propositional logic, the reader is advised to interpret $\vdash$ as deducibility in the system of *intuitionistic relevant* logic. This will enable us to prove a version of Glivenko's Theorem on the way, establishing an important translation of classical logic into intuitionistic relevant logic. Intuitionistic and classical deducibility will be represented by $\vdash_I$ and $\vdash_C$ respectively. Note that

$$\vdash\ \subseteq\ \vdash_I\ \subseteq\ \vdash_C .$$

Define Δ, φ to be $\Delta \cup \{\varphi\}$.

Define

$$\Gamma^{\varphi} =_{df} \Gamma, \varphi \text{ if } \Gamma, \varphi \nvdash \bot;$$
$$\Gamma^{\varphi} =_{df} \Gamma, \neg\varphi \text{ if } \Gamma, \varphi \vdash \bot$$

We call Γ^{φ} the *Lindenbaum expansion* of Γ with respect to φ. It is obtained from Γ by adding φ if it is consistent to do so; otherwise, by adding $\neg\varphi$.

Lindenbaum's Lemma. Lindenbaum expansion preserves consistency; that is,

for all Γ, if $\Gamma \nvdash \bot$ then for all φ, $\Gamma^{\varphi} \nvdash \bot$.

Proof (constructive).[1] The following proof Σ shows the joint inconsistency of the assumptions $\Gamma \nvdash \bot$, $\Gamma^{\varphi} \vdash \bot$ and $\Gamma, \varphi \vdash \bot$:

$$\dfrac{\Gamma \nvdash \bot \qquad \dfrac{\dfrac{\Gamma^{\varphi} \vdash \bot \qquad \dfrac{\Gamma, \varphi \vdash \bot}{\Gamma^{\varphi} = \Gamma, \neg\varphi}}{\Gamma, \neg\varphi \vdash \bot} \qquad \dfrac{\Gamma, \varphi \vdash \bot}{\Gamma \vdash \neg\varphi}}{\Gamma \vdash \bot}\ \text{CUT}}{\bot}$$

[1] If one prefers a non-constructive proof, one can reason more briefly. If $\Gamma, \varphi \nvdash \bot$, then clearly $\Gamma^{\varphi} \nvdash \bot$. If, on the other hand, $\Gamma, \varphi \vdash \bot$, then Γ^{φ} is the result of adding to Γ one of its provable consequences—whence, if Γ is consistent, so will be Γ^{φ}.

Note that the appeal to Cut for *IR* is licit, since the inferred conclusion claims an inconsistency. We now embed Σ as a subproof twice over in the following proof-schema:

$$\dfrac{\begin{array}{c}\underbrace{\Gamma \nvdash \bot ,\ \ \overline{\Gamma^{\varphi} \vdash \bot}^{(2)},\ \ \dfrac{\dfrac{\dfrac{\begin{array}{c}\underbrace{\Gamma \nvdash \bot \ ,\ \ \overline{\Gamma^{\varphi} \vdash \bot}^{(2)} \ ,\ \ \overline{\Gamma, \varphi \vdash \bot}^{(1)}}_{\Sigma} \\ \bot \end{array}}{\Gamma, \varphi \nvdash \bot}(1)}{\Gamma^{\varphi} = \Gamma, \varphi} \qquad \overline{\Gamma^{\varphi} \vdash \bot}^{(2)}}{\Gamma, \varphi \vdash \bot}}_{\Sigma} \\ \bot \end{array}}{\Gamma^{\varphi} \nvdash \bot}(2)$$

Definitions.

Δ is *maximal on atoms* just in case for every atom A, either $A \in \Delta$ or $\neg A \in \Delta$.

Δ is *maximal* just in case for every sentence φ, either $\varphi \in \Delta$ or $\neg\varphi \in \Delta$.

Clearly a set can be maximal on atoms without being maximal. (Example: the set of all atoms.)

Suppose Γ is finite. Then $\Delta; \Gamma \vdash \varphi \ =_{df}$

> there is a (finite) subset Ξ of Δ such that there is a proof of φ whose undischarged assumptions form the set $\Xi \cup \Gamma$.

That is to say, all the sentences in Γ are used as assumptions in at least one proof of φ *modulo* the remaining assumptions Δ. When Γ is a singleton $\{\psi\}$, we write $\Delta; \psi \vdash \varphi$.

Δ *decides* φ just in case either $\Delta \vdash \varphi$ or $\Delta; \varphi \vdash \bot$.

Δ is *decisive* just in case Δ decides every sentence.

A set can be decisive without being maximal. (An example once again would be the set of all atoms—as will be apparent from a result to be proved below.)

Δ is *closed* just in case for every sentence φ, if $\Delta \vdash \varphi$ then $\varphi \in \Delta$.

We now describe a method for expanding a consistent set Δ of sentences so that the expansion is both consistent and maximal on atoms. (To say that Γ expands Δ is just to say that $\Delta \subseteq \Gamma$.)

Let $A_0, A_1, \ldots$ be a list of all atomic sentences. Define

$$\Delta_0 =_{df} \Delta$$

$$\Delta_{n+1} =_{df} \Delta_n^{A_n}$$

$$\Delta^{\mathcal{A}} =_{df} \bigcup_i \Delta_i$$

Obviously

$$\Delta = \Delta_0 \subseteq \Delta_1 \subseteq \ldots \subseteq \Delta_n \subseteq \Delta_{n+1} \subseteq \ldots \subseteq \Delta^{\mathcal{A}}.$$

It is clear from the construction of $\Delta^{\mathcal{A}}$ that it is maximal on atoms. Note that $\Delta^{\mathcal{A}}$ need not in general be maximal, and indeed may even not be closed, since the expansion from Δ to $\Delta^{\mathcal{A}}$ is effected with respect only to *atomic* sentences. Note also that 'the' expansion $\Delta^{\mathcal{A}}$ is uniquely determined by the initial ordering of all atoms. By using different orderings of the atoms one can obtain different expansions. We arbitrarily choose one particular ordering of atoms at the outset, and then exploit the uniqueness of various constructs relative to that initial choice.

Expansion Lemma. If $\Delta \nvdash \bot$ then $\Delta^{\mathcal{A}} \nvdash \bot$.
Proof. Since the Lindenbaum expansion of Δ_n with respect to A_n preserves consistency, and we begin with Δ_0 consistent, it follows by induction that every stage Δ_n is consistent.

Now suppose for reductio that

$$\Delta^{\mathcal{A}} \vdash \bot.$$

Then there is a proof Π of $\bot$ from finitely many premisses drawn from $\Delta^{\mathcal{A}}$. Thus only finitely many of these premisses of Π are not in Δ. Each of the latter premisses is either A_i or $\neg A_i$ for some i; and, if not already in Δ_i, would have been added to Δ_i in order to form Δ_{i+1}. Let A_m be the last atom on the list involved in this way. Then Δ_{m+1} will contain all the premisses of

Π, and thereby be rendered inconsistent. But we have just seen that Δ_{m+1} is consistent. Contradiction. So

$$\Delta^{\mathcal{A}} \nvdash \bot.$$

Lemma. Every maximal consistent set of sentences is closed.
Proof. Suppose Δ is maximal and consistent, and $\Delta \vdash \varphi$. By maximality, either $\varphi \in \Delta$ or $\neg\varphi \in \Delta$. The second case cannot hold—for by $\neg E$ we would have $\Delta \vdash \bot$, contrary to consistency. Thus only the first case can hold; so we are done.

Kalmar's Lemma. For every sentence φ, every set Δ of sentences that is maximal on atoms occurring in φ decides φ. That is: if for every atom A occurring in φ either $A \in \Delta$ or $\neg A \in \Delta$, then either $\Delta \vdash \varphi$ or $\Delta; \varphi \vdash \bot$. (Note: We do not assume, for Kalmar's Lemma, that Δ is consistent.)
Proof. We prove by induction on φ that if Δ is maximal on the atoms occurring in φ, then either $\Delta \vdash \varphi$ or $\Delta; \varphi \vdash \bot$.
Basis. If φ is an atomic sentence A, and Δ is maximal on the atoms occurring in φ, then we have either $A \in \Delta$ or $\neg A \in \Delta$. If $A \in \Delta$, note that A is a proof of A from A; whence $\Delta \vdash A$, and we are done. If $\neg A \in \Delta$, then

$$\frac{\neg A \quad A}{\bot}$$

is a proof of $\bot$ from a member of Δ (namely, $\neg A$) and A; so $\Delta; A \vdash \bot$, and we are done.
Inductive hypothesis. Assume that for every sentence χ simpler than φ, for every set Δ maximal on the atoms occurring in χ, either $\Delta \vdash \chi$ or $\Delta; \chi \vdash \bot$.
Inductive step. Consider complex φ by cases, according to whether φ is of the form $\neg\psi$, $\psi \wedge \theta$, $\psi \vee \theta$ or $\psi \to \theta$. Suppose Δ is maximal on the atoms occurring in φ. Note that Δ will be maximal on the atoms occurring in ψ, and will be maximal on the atoms occurring in θ.

In the case where φ is of the form $\neg\psi$, the reasoning is as follows.

> By inductive hypothesis there are only two cases to consider: (i) $\Delta \vdash \psi$, and (ii) $\Delta; \psi \vdash \bot$.
>
> Case (i). By $\neg E$ we have $\Delta; \neg\psi \vdash \bot$, so we are done.
>
> Case (ii). By $\neg I$ we have $\Delta \vdash \neg\psi$ and once again we are done.

In the case where φ is of the form $\psi \wedge \theta$, the reasoning is as follows.

By inductive hypothesis there are four cases (concerning ψ and θ) to consider. We shall cover all of these cases by considering just the following three: (i) $\Delta \vdash \psi$ and $\Delta \vdash \theta$; (ii) $\Delta ; \psi \vdash \bot$; and (iii) $\Delta ; \theta \vdash \bot$. Since case (iii) is so similar to case (ii), we shall omit the details for case (iii).

Case(i). By $\wedge I$, we have $\Delta \vdash \psi \wedge \theta$, and we are done.

Case (ii). By $\wedge E$, we have $\Delta ; \psi \wedge \theta \vdash \bot$, and we are done.

In the case where φ is of the form $\psi \vee \theta$, the reasoning is as follows.

By inductive hypothesis there are once again four cases (concerning ψ and θ) to consider. We shall cover all of these cases by considering just the following three: (i) $\Delta \vdash \psi$; (ii) $\Delta \vdash \theta$; (iii) $\Delta ; \psi \vdash \bot$ and $\Delta ; \theta \vdash \bot$. Since case (ii) is so similar to case (i), we shall omit the details for case (ii).

Case(i). By $\vee I$, we have $\Delta \vdash \psi \vee \theta$, and we are done.

Case (iii). By $\vee E$, we have $\Delta ; \psi \vee \theta \vdash \bot$, and we are done.

In the case where φ is of the form $\psi \to \theta$, the reasoning is as follows.

By inductive hypothesis there are once again four cases (concerning ψ and θ) to consider. We shall cover all of these cases by considering just the following three: (i) $\Delta \vdash \theta$; (ii) $\Delta ; \psi \vdash \bot$; (iii) $\Delta \vdash \psi$ and $\Delta ; \theta \vdash \bot$.

Case(i). By the 'first half' of the rule $\to I$ in intuitionistic relevant logic, we have $\Delta \vdash \psi \to \theta$, and we are done.

Case (ii). By the 'second half' of the rule $\to I$ in intuitionistic relevant logic, we have $\Delta \vdash \psi \to \theta$, and we are done.

Case (iii). By the proof

$$\dfrac{\psi \to \theta \qquad \begin{array}{c}\Delta \\ \vdots \\ \psi\end{array} \qquad \begin{array}{c}\underbrace{\Delta, \overline{\theta}} \\ \vdots \\ \bot\end{array}}{\bot}$$

using $\to E$, we are done.

A *literal* is an atom or the negation of an atom.

Bivalence Lemma. For every consistent set Θ of literals dealing with all the atoms in φ, either $\Theta \vdash \varphi$ or $\Theta; \varphi \vdash \bot$, but not both.
Proof. Let Θ be a consistent set of literals dealing with all the atoms in φ. It is immediate from Kalmar's Lemma that either $\Theta \vdash \varphi$ or $\Theta; \varphi \vdash \bot$. Now suppose, for reductio, that both $\Theta \vdash \varphi$ and $\Theta; \varphi \vdash \bot$. By cut, $\Theta \vdash \bot$, contrary to hypothesis.[2]

A consistent set of literals is accordingly a perfect proof-theoretic surrogate for the notion of an *assignment of truth-values to atoms*—or, using more general semantic terminology, an *interpretation*. Henceforth we shall call Θ an *interpretation* just in case Θ is a consistent set of literals. If Θ deals with the atoms in φ, then we shall say that Θ *makes φ true* just in case $\Theta \vdash \varphi$, and we shall say that Θ *makes φ false* just in case $\Theta; \varphi \vdash \bot$.

Note the irony: we have 'bivalence' of any interpretation even though the underlying logic of evaluation (here, *IR*) does not contain the classical law of excluded middle, or any of its equivalents. What we have learned is that once one can decide the atoms involved in φ, one can thereby decide φ itself, by means only of inferential moves available within *IR*.[3]

$|\Delta|$ is the set of literals in Δ.

Δ is *satisfiable* just in case some interpretation makes every member of Δ true.

Henkin's Lemma. Every consistent set of sentences is satisfiable.
Proof. Suppose Δ is consistent. Expand Δ to the consistent set $\Delta^{\mathcal{A}}$ that is maximal on atoms. Note that its subset $|\Delta^{\mathcal{A}}|$ is therefore maximal on atoms. Hence by Kalmar's Lemma $|\Delta^{\mathcal{A}}|$ is decisive. Assume now for conditional proof that φ is in Δ. Assume for reductio that $|\Delta^{\mathcal{A}}|; \varphi \vdash \bot$. Since $\phi \in \Delta \subseteq \Delta^{\mathcal{A}}$, this contradicts the consistency of $\Delta^{\mathcal{A}}$. Hence, since $|\Delta^{\mathcal{A}}|$ is decisive, it follows that $|\Delta^{\mathcal{A}}| \vdash \varphi$. Thus we have found an interpretation (namely, $|\Delta^{\mathcal{A}}|$) that makes φ true. But φ was an arbitrary member of Δ. So Δ is satisfiable.

[2] Once again, the appeal to Cut for *IR* is licit, since the inferred conclusion claims an inconsistency.

[3] Actually, *IR* provides more than one needs for the purposes of 'semantic evaluation'. The set of rules exploited in the proof of Kalmar's Lemma make up so-called *truth-table logic*, which is a proper subsystem of *IR*.

Definition. $\Gamma \models \psi$ just in case every interpretation that satisfies Γ makes ψ true.

Double Negation Lemma. $\neg\neg\psi \models \psi$.
Proof. Suppose Θ is an interpretation such that $\Theta \vdash \neg\neg\psi$. Then there is a proof in normal form of $\neg\neg\psi$ from Θ. The proof in question must have the following form:

$$\dfrac{\dfrac{\begin{matrix}\Theta\\ \Pi\\ \psi\end{matrix} \qquad \overline{\neg\psi}^{(1)}}{\bot}}{\neg\neg\psi}{(1)}$$

That is, it must end with a step of $\neg I$. The immediate subproof for this step of $\neg I$ will be normal, reducing $\neg\psi$ to absurdity *modulo* Θ—whence the last step of this subproof must be $\neg E$ with major premiss $\neg\psi$. The proof Π of the minor premiss for that step of $\neg E$ warrants the claim $\Theta \vdash \psi$.

Definition. $\neg\neg\Gamma =_{df} \{\neg\neg\gamma | \gamma \in \Gamma\}$.

Observation. If $\neg\neg\Gamma \vdash \bot$ then (since $\gamma \vdash \neg\neg\gamma$) we have $\Gamma \vdash \bot$. So by contraposition, if Γ is consistent, then $\neg\neg\Gamma$ is too.

Glivenko's Theorem. If $\Gamma \nvdash \bot$ and $\Gamma \models \psi$ then $\neg\neg\Gamma \vdash \neg\neg\psi$.
Proof. Suppose $\Gamma \models \psi$. Then by the definition of $\models$ and the Double Negation Lemma, $\neg\neg\Gamma, \neg\psi$ is unsatisfiable. By Henkin's Lemma, $\neg\neg\Gamma, \neg\psi \vdash \bot$. By consistency of Γ, hence of $\neg\neg\Gamma$, we have $\neg\neg\Gamma; \neg\psi \vdash \bot$. Hence by $\neg I$, $\neg\neg\Gamma \vdash \neg\neg\psi$.

Strong Completeness Theorem. If $\Gamma \models \psi$ then $\Gamma \vdash_C \psi$.
Proof. Suppose $\Gamma \models \psi$. Then $\Gamma, \neg\psi$ is unsatisfiable. By Henkin's Lemma, $\Gamma, \neg\psi \vdash \bot$. By classical reductio (CR), $\Gamma \vdash_C \psi$.

In our treatment thus far we have tried to exploit only the properties of intuitionistic relevant deducibility, so as to obtain Glivenko's Theorem as a by-product of the method of proof of strong completeness that employs Lindenbaum expansion (with respect to atoms) and natural interpretation. If we are willing to exploit the properties of classical deducibility from the start, however, we can prove strong completeness a little more directly.

We say that Γ is ψ-*avoiding* just in case $\Gamma \nvdash_C \psi$. (Hence consistency is simply $\bot$-avoidance.)

Define

$$
\begin{aligned}
{}_{\psi}\Gamma^{\varphi} &=_{df} \Gamma, \varphi \text{ if } \Gamma, \varphi \nvdash_C \psi; \\
{}_{\psi}\Gamma^{\varphi} &=_{df} \Gamma, \neg\varphi \text{ if } \Gamma, \varphi \vdash_C \psi
\end{aligned}
$$

We call ${}_{\psi}\Gamma^{\varphi}$ the ψ-*avoiding Lindenbaum expansion* of Γ with respect to φ.

Lindenbaum's Lemma (ψ-avoidance version). The operation of ψ-avoiding Lindenbaum expansion preserves ψ-avoidance; that is,

> for all Γ, if $\Gamma \nvdash_C \psi$ then for all φ, ${}_{\psi}\Gamma^{\varphi} \nvdash_C \psi$.

Proof. The following proof Σ shows the joint inconsistency of the assumptions $\Gamma \nvdash_C \psi$, ${}_{\psi}\Gamma^{\varphi} \vdash_C \psi$ and $\Gamma, \varphi \vdash_C \psi$:

$$
\dfrac{\Gamma \nvdash_C \psi \qquad \dfrac{\dfrac{{}_{\psi}\Gamma^{\varphi} \vdash_C \psi \qquad \dfrac{\Gamma, \varphi \vdash_C \psi}{{}_{\psi}\Gamma^{\varphi} = \Gamma, \neg\varphi}}{\Gamma, \neg\varphi \vdash_C \psi} \qquad \Gamma, \varphi \vdash_C \psi}{\Gamma \vdash_C \psi}\,\textsc{dilemma}}{\bot}
$$

We now embed Σ as a subproof twice over in the following proof-schema:

$$
\begin{array}{c}
\underbrace{\Gamma \nvdash_C \psi,\quad \overline{{}_{\psi}\Gamma^{\varphi} \vdash_C \psi}^{(2)},\quad \dfrac{\dfrac{\dfrac{\begin{array}{c}\underbrace{\Gamma \nvdash_C \psi \ ,\ \overline{{}_{\psi}\Gamma^{\varphi} \vdash_C \psi}^{(2)} \ ,\ \overline{\Gamma, \varphi \vdash_C \psi}^{(1)}}_{\Sigma} \\ \bot \end{array}}{\Gamma, \varphi \nvdash_C \psi}{}^{(1)}}{{}_{\psi}\Gamma^{\varphi} = \Gamma, \varphi} \qquad \overline{{}_{\psi}\Gamma^{\varphi} \vdash_C \psi}^{(2)}}{\Gamma, \varphi \vdash_C \psi}}_{\Sigma} \\
\dfrac{\bot}{{}_{\psi}\Gamma^{\varphi} \nvdash_C \psi}{}^{(2)}
\end{array}
$$

We can now prove strong completeness more directly as follows.

Define

$$\Gamma_0 =_{df} \Gamma$$

$$\Gamma_{n+1} =_{df} {}_{\psi}\Gamma_n^{A_n}$$

$$\Gamma^{\mathcal{A}} =_{df} \bigcup_i \Gamma_i$$

Expansion Lemma (ψ-avoiding version) . If $\Gamma \nvdash_C \psi$ then $\Gamma^{\mathcal{A}} \nvdash_C \psi$.
Proof. Since the ψ-avoiding Lindenbaum expansion of Γ_n with respect to A_n preserves ψ-avoidance, and we begin with Γ_0 ψ-avoiding, it follows by induction that every stage Γ_n is ψ-avoiding.

Now suppose for reductio that

$$\Gamma^{\mathcal{A}} \vdash_C \psi.$$

Then there is a proof Π of ψ from finitely many premisses drawn from $\Gamma^{\mathcal{A}}$. Thus only finitely many of these premisses of Π are not in Γ. Each of the latter premisses is either A_i or $\neg A_i$ for some i; and, if not already in Γ_i, would have been added to Γ_i in order to form Γ_{i+1}. Let A_m be the last atom on the list involved in this way. Then Γ_{m+1} will contain all the premisses of Π; whence $\Gamma_{m+1} \vdash_C \psi$. But we have just seen that Γ_{m+1} is ψ-avoiding. Contradiction. So

$$\Gamma^{\mathcal{A}} \nvdash_C \psi.$$

Henkin's Lemma (ψ-avoiding version). If $\Gamma \nvdash_C \psi$, then Γ is satisfied by a consistent set of literals that makes ψ false.
Proof. Suppose $\Gamma \nvdash_C \psi$. Expand Γ to the ψ-avoiding set $\Gamma^{\mathcal{A}}$ that is maximal on atoms. By Kalmar's Lemma $|\Gamma^{\mathcal{A}}|$ is decisive. Assume now for conditional proof that $\varphi \in \Gamma$. Assume for reductio that $|\Gamma^{\mathcal{A}}|; \varphi \vdash \bot$. By *ex falso quodlibet*, we have $|\Gamma^{\mathcal{A}}|, \varphi \vdash_I \psi$. Since $\phi \in \Gamma \subseteq \Gamma^{\mathcal{A}}$, it follows that $\Gamma^{\mathcal{A}} \vdash_I \psi$. But $\Gamma^{\mathcal{A}} \nvdash_C \psi$. Contradiction. Hence, since $|\Gamma^{\mathcal{A}}|$ is decisive, it follows that $|\Gamma^{\mathcal{A}}| \vdash \varphi$—that is, $|\Gamma^{\mathcal{A}}|$ makes φ true. But φ was an arbitrary member of Γ. Hence $|\Gamma^{\mathcal{A}}|$ satisfies Γ.

It remains to show that $|\Gamma^{\mathcal{A}}|$ makes ψ false, that is, $|\Gamma^{\mathcal{A}}|; \psi \vdash \bot$. Suppose for reductio that $|\Gamma^{\mathcal{A}}| \vdash \psi$. This contradicts $\Gamma^{\mathcal{A}} \nvdash_C \psi$. So, since $|\Gamma^{\mathcal{A}}|$ is decisive, it follows that $|\Gamma^{\mathcal{A}}|; \psi \vdash \bot$, as required.

Strong Completeness Theorem. If $\Gamma \models \psi$ then $\Gamma \vdash_C \psi$.
Proof. Contrapose Henkin's Lemma.