PHIL153 AUTUMN 1999

INTRODUCTION TO
PROBABILITY, DECISION THEORY AND DATA ANALYSIS.

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## SUMMARY OF LECTURE # 1

There is an important difference between *deductive* and *inductive* reasoning.

1. Deductive reasoning produces proofs of conclusions from premisses (assumptions). These proofs guarantee the transmission of truth from premisses to conclusion. That is, whenever the premisses of a proof are true, its conclusion is true also. Deductive proofs are *sound.* The arguments they establish are *logically valid.* The axiomatic method in mathematics owes its power to the fact that proof always preserves truth. By contrast:
2. Inductive reasoning typically takes one from observations to (general) theories; from particular data to universal generalizations. There is no guarantee that truth will be preserved from the starting point of an inductive inference to its endpoint. The transition from data to theory is sometimes called *inference to the best explanation.*

Mathematics and natural science both aim at the truth. Mathematics aims at necessary truths about abstract structures and objects. Natural science, by contrast, aims at law-like truths[1] about the various natural kinds of physical objects and events forming the causal order in space and time.

## 1 Proof, truth and falsity

Logic serves both mathematics and natural science. Logic does not itself aim at the truth. Rather, it merely transmits truth. It transmits truth from

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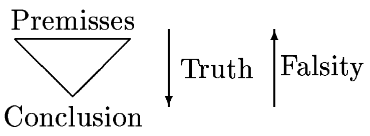
[1] It is a matter of contemporary debate whether these laws are *metaphysically necessary*, even though not *logically necessary*; and a problem for contemporary philosophers to exlain whether, and if so how, law-like truths differ from accidental truths. We cannot go into these issues here.

the premisses to the conclusion of a logically valid argument. It also re-transmits falsity. That is, it re-transmits falsity from the conclusion to the premisses of a logically valid argument. Logic provides *proofs* of logically valid arguments. I shall represent a proof schematically as a triangle, with its premisses at the top and its conclusion at the bottom:



It is the task of logic to provide means of constructing such proofs for any valid argument. This is done by means of *rules of inference* that govern the so-called *logical words* (words like *not, and, or, if ... then ..., some, all, ... is identical to ...*) in the premisses and conclusion of the argument. The formal symbols often used for these logical words are as follows:

| | |
|---|---|
| not | $\neg$ |
| and | $\wedge$ |
| or | $\vee$ |
| if... then... | $\rightarrow$ |
| some | $\exists$ |
| all | $\forall$ |
| is identical to | $=$ |

There are of course alternative notational conventions allowing the use of different symbols than those displayed above.

When an argument is valid—that is, when it transmits truth from its premisses to its conclusion—we say that the premisses *logically imply* the conclusion, or that the conclusion is a *logical consequence* of the premisses. When we have a proof of the conclusion from the premisses, we say that we have *inferred* or *deduced* the conclusion from the premisses. In that case we shall write

$$\textit{Premisses} \vdash \textit{Conclusion.}$$

The symbol $\vdash$ is called the *turnstile*, and is used to claim the *existence of a proof* showing that the conclusion follows from the premisses.

It is the task of so-called *deductive* logic to match validity with deduction: that is, to provide any valid argument with a proof.

The orthodox view is that *whether* an argument is logically valid depends only on the structure of its premisses and its conclusion, and *not* on the existence of a proof making clear that it is indeed valid. Thus one could give an argument consisting barely of its premisses and conclusion, with no hint as to how to negotiate the route from the former to the latter; and assume that there is a fact of the matter as to whether truth really is transmitted from the premisses to the conclusion. A proof would be welcome in *establishing* that fact, and making it evident or certain; but, on this view, the proof would play no role in *constituting* the fact that the argument was valid. A less orthodox, opposed view is that the validity of any argument is indeed *constituted* by the existence of a proof leading from its premisses to its conclusion. So long, however, as we concern ourselves only with systems of logic that do provide proofs for every valid argument, we shall not need to take a stand on this issue for the purposes of the ensuing discussion.

When logicians and philosophers talk about theories, they usually intend the word in an idealized sense: a theory is a set of statements that is *logically closed*, that is, a set already containing all statements that follow logically from it. Thus a theory will be infinite, even is only because every statement $P$ logically implies its own double negation: $P \vdash \neg\neg P$. If $T$ is a theory, then a set of axioms for $T$ is any subset of $T$ whose logical closure is $T$ itself. We say that $T$ is *finitely axiomatizable* just in case $T$ has a set of axioms that is finite.

A special case of proof is that of *disproof*: when the conclusion is something absurd. This could either be a contradiction of the form *P and not-P*, or a statement involving *contraries*, such as 'This is solidly green all over now and this is solidly red all over now'. We shall use the so-called *absurdity* symbol $\perp$ to register situations of this kind. $\perp$ cannot be true. $\perp$ is always false. (It is an interesting question whether contrariety is conceptually prior to the notion of sentential negation, and therefore prior also to the formal notion of contradiction afforded by negation and conjunction. Again, however, we cannot pursue this question here.)

When a disproof is given:

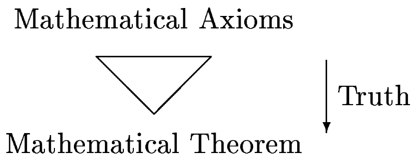
we know that the premisses cannot all be true. At least one of them must be false. We say that they have, collectively, been *reduced to absurdity* by means of the (dis)proof. They have been shown to be *jointly inconsistent.*

*Consistency* of a theory is *necessary* but not *sufficient* for its truth. A remarkable feature of the crazy stories or world-views of some highly intelligent paranoid schizophrenics is that they are often consistent. As one challenges their statements, they shore them up with more and more bizarre ones. Their story spins on, always coherent, but—the normal observer feels—more and more bizarrely. This should suffice to show that mere consistency, or coherence, is not enough for truth!

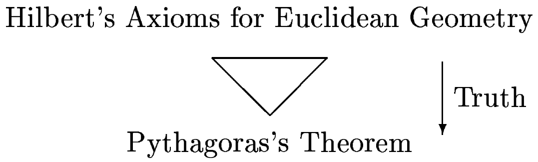
## 2 The foundational method in mathematics

What, then, *is* enough for truth? Let us suppose, quite generally, that we have direct access to *some* truths. How do we get to others? The answer depends on whether we are doing mathematics or natural science.

Mathematics starts with *axioms*, and proves *theorems* from them—the paradigm of a *foundational* method. Ideally, the axioms are safe, obvious, simple, certain; while the theorems will be of varying depth, interest, profundity and importance. One thing of which we are sure, however, is this: any properly constructed proof shows that *if* the axioms it uses are true, *then* the theorem which is its conclusion is true also. The proof is a logical passage. Imagine it starting with the axioms at the *top*, and heading *down* to the theorem at the *bottom.* The proof guarantees that truth will be transmitted *from* the mathematical axioms *to* the mathematical theorem. That is: *if* the axioms are true, *then* the theorem will be true also. The proofs themselves provide no guarantee of the truth of the axioms. The axioms have to be intuited as true, or assumed to be true. The proof then gives a logical guarantee that the (assumed) truth of the axioms will *transmit to* the conclusion. This means that the conclusion will be at least as secure as the axioms; but still depends on those axioms for its truth, in the sense that it is the assumed truth of the axioms, plus the logically correct structure of the proof, that furnishes our grounds for believing the conclusion of the proof to be true. Indeed, we are usually *certain* about axioms of mathematics; that is why we can use them as a foundation for the rest of our mathematical knowledge:

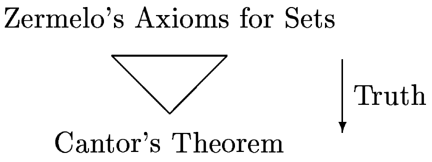


An example of this scheme would be



Pythagoras's Theorem is the famous claim that the square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the other two sides. Hilbert's axioms state basic properties of points, lines, incidence and betweenness. An example of such an axiom is the claim that any two points determine a unique line. From such simple and obvious-sounding starting points, logical proof enables one to reach much less obvious-seeming theorems, such as Pythagoras's Theorem. This is one of the great puzzles in the philosophy of logic: how can deductive reasoning apparently 'increase information' in this way, if the truth of the conclusion of a deductive proof is *logically guaranteed* by the (assumed) truth of its premisses? The logical guarantee is a matter of *necessity.* The increase in information is a matter of *fruitfulness.* The problem, in a nutshell, is: how can one reconcile the necessity of deduction with its fruitfulness?

Another example of our schema for mathematical reasoning would be



Cantor's Theorem states that every set has strictly more subsets than it has members. Now the *power set* of any set is the set of all its subsets. So, put another way, Cantor's Theorem says that there is no one-one mapping from the power set of any set onto the set itself. Zermelo's axioms for set theory include simple and obvious-seeming claims such as: given any two sets, one can form a set consisting of just those two sets; given any set, one can form the set of all members of its members; etc. Cantor's Theorem shows how very powerful the *Power Set Axiom* is; that is, the axiom that says that if a set $X$ exists then so does the power set of $X$. The power set of $X$ will contain strictly more members than $X$ itself. That means that if one postulates the existence of any infinite set such as the set $\{0, 1, 2, \ldots\}$ of natural numbers, then one knows from Cantor's Theorem that its power set will be 'even more infinite'! So, at a stroke, there will be infinitely many infinities!—all this from some very innocuous and obvious-seeming axioms about sets.

Logical proof somehow rearranges the import of the mathematical axioms to reveal surprising, enlightening and profound further results about the mathematical realm being studied. And it does so without going 'beyond' the axioms taken as the starting points of the inquiry. It simply wrings from those axioms new and informative content that was 'already there', so to speak, latent within them. The new truths, arrived at as the conclusions of proofs, are *guaranteed* by the truth of those simple axioms—necessity plus fruitfulness, again.

## 3 Logic and the axiomatic method in mathematics

First, a proof-system gives a foundation to knowledge of certain kinds (such as mathematics), and serves to make truth more certain. In mathematics one starts with various simple, obvious, certain truths called axioms (such as the statement that if you add 0 to any number you get the same number). Then, by applying logic, one advances to more difficult and complicated truths (usually called theorems) that follow logically from the axioms. An example of a theorem is the statement that every non-zero number divides into any other number a unique number of times, and exceeds the (unique) remainder. An example of a statement that has not yet been shown to be false, and has not yet been proved true, is Goldbach's Conjecture: every even number greater than two is the sum of two prime numbers.

Logicians are involved in the study of statements that may be undecidable on the basis of the axioms at one's disposal. A statement is undecidable in a given theory if the theory contains neither the statement nor its negation. (Remember that a theory is logically closed, that is, it contains every statement that it implies.) So an undecidable statement for a given *axiomatized* theory is one that cannot be proved from, nor be proved to be inconsistent with, the axioms of that theory. Goldbach's Conjecture is at present undecided, but no-one knows whether it is undecidable (on the basis of our present axioms for number theory). Two celebrated principles in the foundations of mathematics, namely the Continuum Hypothesis and the Axiom of Choice, are known to be undecidable within the system of Zermelo-Fraenkel set theory, which most mathematicians nowadays take as their working basis. We also know that arithmetic cannot be completely axiomatised: for any proposed system of axioms, there will be true statements about numbers that cannot be decided by the axioms. This famous result is due to the logician Gödel, and is known as his first incompleteness theorem for arithmetic. His second incompleteness theorem states that we cannot prove the consistency of arithmetic unless we use a system strictly stronger than arithmetic itself. The first incompleteness theorem has been claimed by some philosophers (such as J.R.Lucas) to have profound implications for the question whether our minds transcend the limitations in principle on the operation of any machine.[2]

The second incompleteness theorem also has implications for the foundations of knowledge. It put paid to the so-called Hilbert programme in the foundations of mathematics. Hilbert had aspired to prove the consistency of all mathematics by purely finitary and formalistic means.

## 4 Logic and scientific method

The second application of a proof-system is in theoretical science. Here we test our theories against the evidence. Theories consist of theoretical or explanatory hypotheses. Sometimes these are of very high degrees of generality . They may speak of all things of such-and-such a natural kind; or they may say that some quantity, like energy or momentum, is always conserved in situations of such-and-such a kind. Sometimes the theoretical hypotheses invoke theoretical or unobservable entities such as quarks, photons and electrons. We entertain only those hypotheses that are based on or supported by the evidence. Remember it is controversial whether there is any such thing as an inductive logic that would validate the transition from the evidence to the hypotheses. The exact nature of this 'support' is a

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[2] See A. R. Anderson, ed., *Minds and Machines*, Prentice-Hall Contemporary Perspectives in Philosophy Series, Englewood Cliffs, 1964.

deep problem in the theory of knowledge and the philosophy of science. It is known as the problem of induction. So-called inductive logic (if there is such a thing) would allow us to pass from the bits of evidence, as premisses, to our general hypotheses, as conclusions. The pattern would be this:

Various bits of observational evidence
as premisses

===================== (inductive leap)

Hypothesis as conclusion

The evidence would at best make the conclusion highly probable; it would not provide a logical guarantee of its truth. The inductive leap is not deductively valid. Good theoretical hypotheses, as we saw above, do not just recapitulate the evidence and say no more. Rather, they 'go beyond' the evidence on which they are based. For they imply predictions about the behaviour of systems, or the distribution of characteristics in a population, or the motions of heavenly bodies, etc. that have not yet been observed or measured. In doing so, they allow us to anticipate the course of our future experience, and to exercise some control over future events. Note that the predictions are deductively implied by the hypotheses. Deductive logic would yield proofs of the downward passages in the following schema:

$$\begin{array}{l} \text{Mathematical axioms} \\ \qquad\quad\vdots \\ \underbrace{\text{Mathematical theorems, Scientific hypotheses, Boundary conditions}} \\ \qquad\qquad\qquad\qquad\qquad\quad\vdots \\ \qquad\qquad\qquad\qquad\quad\text{Predictions} \end{array}$$

But our observations and measurements may conflict with, or disagree with, or contradict our predictions:

$$\begin{array}{l}
\text{Mathematical axioms} \\
\qquad\quad\vdots \\
\underbrace{\text{Mathematical theorems, Scientific hypotheses, Boundary conditions}} \\
\qquad\qquad\qquad\qquad\qquad\quad\vdots \\
\qquad\qquad\qquad\qquad\quad\underbrace{\text{Predictions, Observations/measurements}} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\vdots \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\perp
\end{array}$$

In such a situation we are (rationally) forced to revise our theoretical hypotheses. The problem of how to revise one's theories in the light of new conflicting evidence has recently become very important in computing science. So-called expert systems with data bases are programmed to simulate human knowledge and principled understanding of various aspects of the world (oil-drilling; the legal system; automatic piloting). They have to update their data, and sometimes revise their principles if they are to perform satisfactorily. To do so (that is: for us to be able to make them do so) we have to understand the logic of theory change, or theory dynamics, as it is now known. We may, of course, refuse to revise our theoretical hypotheses in the face of conflicting evidence. We may decide instead to blame our microscopes or our telescopes or our measuring apparatus ('bad data'). Or we may doubt whether the boundary conditions for the experiment were properly controlled ('bad control'). Or we may just hang on to our hypotheses at all costs, because they are the best we have, and choose to disregard the new evidence ('it was a fluke'; 'unrepeatable result'). The problem of where to lay the blame in a situation like this—whether to fault the theoretical hypotheses or the 'data' or other auxiliary assumptions—is known as the Quine-Duhem problem, after the two philosophers who emphasised its importance for methodology.