Pergamon

Stud. Hist. Phil. Sci., Vol. 29, No. 4, pp. 589–621, 1998

Printed in Great Britain
0039–3681/98 $19.00+0.00

Kant on the 'Symbolic Construction' of Mathematical Concepts

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In the chapter of the *Critique of Pure Reason* entitled 'The Discipline of Pure Reason in Dogmatic Use', Kant contrasts mathematical and philosophical knowledge in order to show that pure reason does not (and, indeed, *cannot*) pursue philosophical truth according to the same method that it uses to pursue and attain the apodictically certain truths of mathematics. In the process of this comparison, Kant gives the most explicit statement of his critical philosophy of mathematics; accordingly, scholars have typically focused their interpretations and criticisms of Kant's conception of mathematics on this small section of the *Critique*.

Many of Kant's most important and familiar claims pivot on the conception of mathematics that he articulates in the 'Discipline'. Specifically, Kant's arguments for transcendental idealism depend on his claim that mathematical knowledge is synthetic *a priori*; this claim follows from his understanding of mathematical knowledge as based on the construction of concepts in intuition. Thus, a successful interpretation of the 'construction' of mathematical concepts is an important component of understanding and appreciating the *Critique*, and a valuable first step toward re-evaluating Kant's contribution to the philosophy of mathematics.

A full understanding of Kant's notion of the 'construction' of mathematical concepts must confront two notoriously obscure (but much discussed) passages from the 'Discipline', at A717/B745 and A734/B762, the only two passages of the *Critique* in which Kant mentions algebra. Here Kant claims that algebra 'achieves by a symbolic construction equally well what geometry does by an ostensive or geometrical construction . . .' (Kant, 1998, A717/B745). In what follows I will show that the common interpretation of these important passages, and thus of Kant's philosophy of mathematics in general, is flawed. The major Kant commentators have failed to give a reading that is informed by the state of early modern mathematical practice; rather, their reading has been informed by a more recent concep-

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Received 3 March 1997; in revised form 17 October 1997.

PII: S0039-3681(98)00023-5

tion of mathematics that is inadequate for understanding Kant's claims about geometry, arithmetic, and, in this case, algebra. I hope to present a more compelling reading of our passages by taking a contextualist approach, seeking to understand and elucidate Kant's claims about symbolic construction on the basis of the conception of algebra he would have gleaned from his immersion in the textbook mathematics of the eighteenth century.[1]

In the first section, I introduce the passages in question, and briefly examine the context in which Kant has occasion to mention algebra. Then, in the second section, I review the secondary literature and outline what I take to be the common, misleading assumptions made by those scholars who have offered interpretations, or reconstructions, of these particular passages. I also show how the misreadings that result are neither motivated by nor consistent with Kant's view of mathematics as stated elsewhere.

In the third section, I describe an area of early modern mathematics called 'the application of algebra to geometry' in which the method of 'constructing equations' was demonstrated and applied. This area of elementary mathematics was common to college-level textbooks of the seventeenth and eighteenth centuries; specifically, it was included in the popular textbooks by Christian Wolff that Kant used to teach courses in mathematics.[2] My examination of early modern mathematical practice serves to clarify the relationships between the elementary mathematical disciplines of arithmetic, geometry and algebra; in particular, I show that algebra was not conceived as a separate mathematical discipline with its own object of investigation. Rather, algebra was conceived as a method of reasoning about the objects of arithmetic and geometry, and was thus used as a tool for solving arithmetic and geometric problems.[3]

In the fourth section, I offer a reading of Kant's remarks on algebra that is inspired by and consistent with the mathematical practice with which he was engaged. In particular, I use the discussion of Wolff's application of algebra to the problems of arithmetic and geometry to illustrate Kant's notion of the 'symbolic construction' of algebraic concepts. I conclude that Kant did not intend to draw a

[1]The scope of this paper will not allow me to interpret fully Kant's claim that mathematical knowledge relies on the construction of mathematical concepts, nor the related and more general issue of the role of diagrams in mathematical demonstration. I am here concerned to explicate Kant's notion of symbolic construction, without completing the more formidable task of explicating Kant's notion of construction of concepts in general. Consequently, we must take as a background assumption to the arguments that follow that, from his investigation of Euclidean geometry, Kant concluded that the ostension of constructed geometric figures is a necessary feature of geometrical cognition.

[2]For a discussion of Kant's teaching schedule and curriculum, as well as a list of the textbooks he used to teach mathematics, see the introduction to Martin (1985).

[3]I want to emphasize that in this paper I am concerned with the eighteenth-century elementary textbook mathematics that directly influenced Kant's conception of mathematics as it is presented in the *Critique*, first published in 1781. It is a separate and interesting question to ask whether and how Kant's critical philosophy of mathematics could account for late eighteenth-century developments in the theory of algebraic equations that led to Galois theory and group theory in the nineteenth century, and, ultimately, to our modern structural conception of abstract algebra. Because algebraic practice was conceived with respect to its geometric applications until at least 1870 (van der Waerden, 1985, p. 137), I would speculate that Kant's philosophy of algebra as I will interpret it remained viable for at least a century.

strict distinction in kind between symbolic construction on the one hand and ostensive or geometrical construction on the other. Rather, a 'symbolic' construction is, for Kant, that which symbolizes an ostensive construction. Finally, in the fifth and final section, I conclude by explaining the importance of 'symbolic construction' to both the early modern algebraist and the critical philosopher.

1.

The passages at issue fall in the section of the *Critique* in which Kant intends to 'discipline' pure reason with respect to its methods of attaining transcendental knowledge. Kant's first task in this regard is to determine 'whether the method for obtaining apodictic certainty that one calls *mathematical* in the latter science (i.e., pure reason in its mathematical use) is identical with that by means of which one seeks the same certainty in philosophy, and that would there have to be called *dogmatic*' (Kant, 1998, A713/B741). It is this question which prompts Kant to compare the methods of attaining philosophical and mathematical knowledge, and to elaborate his claim that mathematical cognition is attained by the construction of concepts in intuition.

To illustrate the power of the mathematical method, Kant cites the familiar theorem of Euclidean geometry which establishes the relation between the angle sum of a triangle and a right angle;[4] he shows that the philosophical method of analysing general concepts is inadequate in the face of such a task. In contrast, the mathematician's ability to construct the concepts whose relations are under investigation in any particular theorem allows the geometric figures called for to be drawn in accordance with the general conditions for their construction, enabling cognition of those properties of the concept which, though not contained in its (discursive) definition, do necessarily belong to it.[5] The drawn geometric figures enable the mathematician to consider universal concepts *in concreto*, that is, in the form of individual intuitions that nevertheless serve to express universal concepts by virtue of the rule-governed act which legislates their construction; by contrast, the philosopher, reasoning discursively, has access to the universal concepts only *in abstracto* and can thereby predicate of such concepts only what is already contained in them.

This brings us to the first passage at issue. Almost as an aside to his discussion of the mathematical method, which has thus far taken all of its examples from Euclidean geometry, Kant says:

[4]Book I, Proposition 32 of Euclid's *Elements*, which reads 'In any triangle, if one of the sides be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of the triangle are equal to two right angles.' (Euclid, 1956, p. 316)

[5]These general conditions for construction are encompassed by the definitions, postulates, and axioms of Euclidean geometry and generally warrant the use of straight-edge and compass constructions. According to Kant, the 'drawings' can be produced either in pure intuition *via* the productive imagination, or empirically on paper, in both cases *a priori*. The details of Kant's account of geometric construction, and the role of the *schema* of a mathematical concept, cannot be pursued here. It is sufficient for my purposes here to recognize simply that Kant takes the construction of the triangle and auxiliary lines in the proof of Euclid's I.32 to exemplify the geometric construction of concepts.

> But mathematics does not merely construct magnitudes (*quanta*), as in geometry, but also mere magnitude (*quantitatem*), as in algebra, where it entirely abstracts from the constitution of the object that is to be thought in accordance with such a concept of magnitude. In this case it chooses a certain notation for all construction of magnitudes in general (numbers), as well as addition, subtraction, extraction of roots, etc. and, after it has also designated the general concept of quantities in accordance with their different relations, it then exhibits all the procedure through which magnitude is generated and altered in accordance with certain rules in intuition; where one magnitude is to be divided by another, it places their symbols together in accordance with the form of notation for division, and thereby achieves by a symbolic construction equally well what geometry does by an ostensive or geometrical construction (of the objects themselves), which discursive cognition could never achieve by means of mere concepts. (Kant, 1998, A717/B745)

It is Kant's notion of 'symbolic construction', introduced here to account for the algebraic method, that commentators have found so puzzling and opaque.

Looking for further evidence of Kant's view of 'symbolic construction', scholars have typically analyzed the preceding passage alongside his second (and only other) mention of algebra in the *Critique*. Continuing to compare the methods of mathematics and philosophy in their search for certain knowledge, Kant identifies several unique aspects of the mathematical method that distinguish it from the philosophical: definitions, axioms, and demonstrations. With regard to the last, Kant claims that 'only mathematics contains demonstrations, since it does not derive its cognition from concepts, but from their construction, i.e., from the intuition which can be given *a priori* corresponding to the concepts' (Kant, 1998, A734/B762). Implying that such a claim is obvious with respect to the familiar demonstrations of Euclidean theorems, Kant continues:

> Even the way algebraists proceed with their equations, from which by means of reduction they bring forth the truth together with the proof, is not a geometrical construction, but it is still a characteristic construction, in which one displays by signs in intuition the concepts, especially of relations of quantities, and, without even regarding the heuristic, secures all inferences against mistakes by placing each of them before one's eyes. (Kant, 1998, A734/B762)

Care must be taken in attempting to interpret any thinker's position on the basis of two brief passages. Nevertheless, Kant's philosophy of mathematics plays a crucial role in his critical philosophy, and a clear understanding of his notion of mathematical construction would do much to elucidate his general epistemology. But a clear understanding of Kant's view of mathematical construction requires that we arrive at a satisfactory interpretation of 'symbolic' or 'characteristic' construction based on the two passages cited. That is the task of what follows.

2.

Many contemporary Kant scholars, both critical and sympathetic, have used the passages cited to develop readings of what Kant's philosophy of algebra might have been.[6] They do so, I think, to achieve a balance on Kant's behalf: Kant's arguments throughout the *Critique* draw heavily on examples from Euclidean geometry, but rarely from disciplines outside Euclidean geometry, for evidence of our mathematical cognition in general. By suggesting ways in which Kant's philosophy of geometry is paralleled by a corresponding philosophy of arithmetic and algebra, Kant's commentators can ascribe to him a complete philosophy of elementary mathematics and vindicate his claim to have provided a theory of the mathematical method in general. Moreover, this theory would account for the construction in intuition of all mathematical concepts, not just the obviously constructible concepts of Euclidean geometry. Alternatively, some commentators analyze the same passages in an attempt to show that exploiting Kant's philosophy of geometric cognition to account for arithmetic or algebraic cognition is hopeless, thus proving that Kant had no mature philosophy of mathematics in general at all.

C. D. Broad, in his widely read article 'Kant's Theory of Mathematical and Philosophical Reasoning' (Broad, 1941), takes an extremely pessimistic view of the matter. He writes that '[Kant's theory of mathematics] was evidently made up primarily to deal with geometry, and was then extended forcibly to deal with arithmetic and algebra' (Broad, 1941, p. 5), and concludes that 'It seems to me, then, that Kant has provided no theory whatever of algebraical reasoning . . .' (Broad, 1941, p. 23).

While more recent scholars have not often taken such a dim view of Kant's brief comments on algebraic cognition, they have nevertheless employed the same or similar assumptions as those at work in Broad's argument. Broad assumes that the 'x's and y's of algebra . . . are purely arbitrary symbols for any number taken at random . . .' (Broad, 1941, p. 22). Moreover, he takes the inscription of such symbols to be the pure constructions in intuition on which Kant's theory of algebra must rely: '[In Kant's] account of algebra he admits the merely symbolic and non-instantial construction of concepts. Of course, these constructs themselves will be spatio-temporal, since the symbols will consist of perceived or imagined marks.' (Broad, 1941, p. 24) I will consider these two assumptions separately.

First, Broad assumes that whenever letters are used in an algebraic context, they designate either constant or variable numeric values, depending on whether they are selected from the beginning or the end of the alphabet respectively. This assumption, which seems natural given our current conception of elementary algebra, assimilates Kant's phrases 'magnitude in general', 'mere magnitude', and 'general concept of magnitude' to our notion of a variable ranging over infinitely many possible numeric values. Underlying this assumption is the deeper supposition that algebra is simply a generalized arithmetic: while arithmetic is the mathematics of determinate numeric quantities, algebra is that of indeterminate numeric

[6]Included among those who have offered interpretations of Kant's notion of 'symbolic construction' are Broad, Brittan, Friedman, Hintikka, Kitcher, Martin, Parsons, Thompson, and Young. See Broad (1941), Friedman (1992), Martin (1985), and, especially, the anthology of articles edited by Posy (1992).

quantities. Again, such a supposition assimilates Kant's phrases 'general arithmetic' and 'universal mathematics' to our common understanding of elementary algebra.[7]

Second, Broad assumes that the 'symbolic construction' ascribed by Kant to algebraic cognition is a construction *of* or *out of* symbols in the same way that a geometric construction is the construction of a geometric object, such as a triangle, out of other geometric objects, straight lines. That is, Broad supposes that the construction of algebraic concepts, which he takes to be variable numeric quantities, consists in the construction of a symbol of such, for example 'x'. On this view, the inscription of the symbol 'x' would exhibit the concept of a variable numeric quantity in intuition just as, for Kant, a three-sided plane figure exhibits the concept of a triangle.

More recent commentators employ similar assumptions to Broad's in their analyses of Kant's view of algebraic cognition. The legendary interpretive disagreement between Jaakko Hintikka and Charles Parsons over the status of the Kantian notion of *Anschauung*[8] has prompted each of them to provide an account of Kant's view of the intuitions involved in mathematical cognition in general, and in algebraic cognition in particular. Despite arriving at very different readings of our two passages, they nevertheless employ similar assumptions in the arguments for their respective positions. Hintikka, for example, writes:

> If we can assume that the symbols we use in algebra stand for individual numbers, then it becomes trivially true to say that algebra is based on the use of intuition, i.e., on the use of representatives of individuals as distinguished from general concepts. After all, the variables of elementary algebra range over numbers . . . (Hintikka, 1992, p. 26)

He concludes that when two algebraic symbols 'a' and 'b' are combined *via* an algebraic operator such as ' + ', a new individual number is represented by the expression 'a + b'. On Hintikka's view, Kantian construction amounts to the introduction of intuitions that represent new individuals, and in the algebraic case, 'a + b' is the constructed object. Likewise, Parsons writes:

> The algebraist, according to Kant, is getting results by manipulating *symbols* according to certain rules, which he would not be able to get without an analogous intuitive representation of his concepts. The 'symbolic construction' is essentially a construction with *symbols* as objects of intuition. (Parsons, 1992, p. 65)

[7]Kant does not use these phrases in the *Critique of Pure Reason*; they occur in his pre-critical 'prize essay' entitled 'Inquiry concerning the distinctness of the principles of natural theology and morality', reprinted in Kant (1992). Though some commentators have looked to this essay for support of their interpretations of Kant's view as it is articulated in the *Critique*, I think Kant's pre-critical and critical views must be treated separately. Consequently, I will provide a reading of the 'Inquiry' elsewhere.

[8]This debate concerns Hintikka's interpretation of Kantian intuition as any representative of an individual, and Parsons' rejection of this 'singularity' criterion in favor of an 'immediacy' criterion. Parsons argues that while immediacy implies singularity, the converse does not hold. In this paper, I will make no effort to adjudicate between Hintikka and Parsons with respect to their accounts of *Anschauung*, as my concern is only with certain of their key assumptions and sub-arguments, and not with their broader conclusions. For the literature of the debate itself, as well as various articles on issues raised by the debate, see the anthology of articles edited by Posy (1992).

Thus, despite arguing against one another for directly opposing interpretations of the primary function of intuition in mathematics, Hintikka and Parsons nevertheless proceed from the common assumption that by 'symbolic construction' Kant intended to describe a construction built from algebraic symbols. Moreover, their analysis of Kant's position follows from considering the rules of algebraic reasoning by analogy to numerical rules of calculation.[9]

In *Kant and the Exact Sciences*, Michael Friedman presents the most recent analysis of Kant's philosophy of mathematics, making 'contextualism' a guiding principle of his project:

> Kant's philosophical achievement consists precisely in the depth and acuity of his insight into the state of the mathematical exact sciences as he found them, and, although these sciences have radically changed in ways that were entirely unforeseen (and unforeseeable) in the eighteenth century, this circumstance in no way diminishes Kant's achievement . . . My aim throughout is to show that and how central aspects of the Kantian philosophy are shaped by—are responses to—the theoretical evolution and conceptual problems of contemporary mathematical science. (Friedman, 1992, pp. xii–xiii)

True to his aim, Friedman assesses Kant's theory of mathematical cognition, and his specific claims about symbolic construction and algebraic cognition, within the context of eighteenth-century mathematical practice and offers a rich and provocative view. Nevertheless, Friedman inherits crucial assumptions from the Broad tradition that undermine his new reading.[10]

First, Friedman claims that even though Kant mentions only algebra in the passage at A717/B745, 'it is likely that both arithmetic and algebra are to be included under symbolic or characteristic construction'; moreover, he claims that, for Kant, 'algebra appears to be a kind of arithmetic: "the general arithmetic of indeterminate

[9]Kitcher, Thompson, and Young make similar assumptions in their articles Kitcher (1992), Thompson (1992), and Young (1982, 1992). For example, Thompson argues that, for Kant, the constructions of algebra are 'spatial representations' of numbers and numerical relations, i.e. written numerals and formulas (Thompson, 1992, p. 97). Kitcher writes: 'The theory of "symbolic construction" for algebra only amounts to the weak claim that algebra is "intuitive" in being able to operate with signs.' (Kitcher, 1992, p. 119) And Young, claiming that arithmetic calculation exemplifies symbolic construction, writes that '. . . the numeral string can be said to provide a *symbolic* construction of the corresponding arithmetical concept. The column of numeral strings that we use in calculation can likewise be said to provide a symbolic construction of the concept of the sum that we seek to determine.' (Young, 1982, p. 23) In a later paper, he makes a similar point: 'Kant introduces the notion of symbolic construction only in his discussion of algebra. Like Parsons, however, I believe it is legitimate to extend the notion and to describe both the use of numerals, in calculation, and the use of formulae in logic as involving symbolic construction.' (Young, 1992, p. 173)

[10]Again, I do not intend to evaluate Friedman's book as a whole, nor his overall interpretation of Kant's philosophy of mathematics. I hope only to isolate some key assumptions which situate his view, generally speaking, in the tradition inaugurated by Broad. For a discussion and assessment of Friedman's analysis of Kant's engagement with early modern mathematics and science, see Hatfield (1996), especially pp. 122–30.

magnitudes"' (Friedman, 1992, p. 108).[11] But he distinguishes himself from the other commentators by, for example, rejecting the inference that algebra simply generalizes over all the particular numbers of arithmetic (Friedman, 1992, pp. 108–9). His alternative suggestion is to suppose that for Kant '"General arithmetic (algebra)" goes beyond arithmetic in the narrower sense, not by generalizing over it, but by considering a more general class of magnitudes.' (Friedman, 1992, p. 109) So, arithmetic deals with magnitudes which have a determinate ratio to unity, that is, rational magnitudes, and algebra with magnitudes which have an indeterminate ratio to unity, that is irrational (or incommensurable), magnitudes. Consequently, arithmetic would correspond to the theory of numerical magnitudes in Euclid's *Elements* (Books VII–IX) and algebra to the Euclidean/Eudoxean theory of ratio and proportion (Book V) (Friedman, 1992, p. 110).

Friedman then claims that algebra, the theory of ratios, is able to measure incommensurable magnitudes as precisely as necessary: '[algebra] allows us to find a definite rule of approximation by numbers (including fractions), a rule of approximation which can be made as accurate as one wishes' (Friedman, 1992, p. 112). As a 'rule of approximation' he has in mind the decimal expansion, or infinite series of fractions, that approximates any given irrational magnitude.

Finally, after identifying the procedure of successive iteration as that which underlies Kant's general theory of the construction of mathematical concepts, Friedman uses his interpretation of the historical roles of arithmetic and algebra to give a reading of 'symbolic construction'. He writes:

> There are actually two distinguishable, although closely related, aspects to symbolic construction. On the one hand, in finding the magnitude of anything we will employ the successive progression underlying the number series: either by generating a whole number or fraction in a finite number of steps or by generating an infinite approximation to an irrational number. On the other hand, however, successive iteration is also employed in the mere manipulation of signs in algebraic formulas: such 'operation of a calculus' is also an iterative, step by step procedure. (Friedman, 1992, pp. 119–120)

Even though he does not exploit the simplicity of the assumptions made by Broad and others, Friedman's view is problematic in several similar respects. His assumption that Kant considers algebra to be a kind of arithmetic still leads him to construe the role of algebra too narrowly, as chiefly involved in numerical calculation. Moreover, though he does not isolate the inscripted algebraic symbol as Kant's intended construction, he sees the formal manipulation of such symbols as a central feature

[11] In his reading of A717/B745, Friedman puts quotation marks around 'general arithmetic' and 'indeterminate magnitude' (Friedman, 1992, p. 109). However, these phrases do not occur in the Kemp-Smith translation of the *Critique* (the source that Friedman cites when he quotes the *Critique* in English), but in the 'Inquiry' and letters, to which Friedman refers for textual support of his reading (Friedman, 1992, p. 108). It is unclear whether Kant's arguments in the pre-critical 'Inquiry' can necessarily be employed to facilitate a reading of A717/B745. The phrases Kant uses in the *Critique* include 'bloße Größe', translated as 'magnitude as such' by Kemp-Smith and as 'mere magnitude' by Guyer and Wood, and 'Größe überhaupt', translated as 'magnitude in general' by Kemp-Smith and Guyer and Wood; these phrases will be clarified below.

of 'symbolic construction'.[12] Ultimately, despite its refinements, Friedman's view continues to analyze algebraic cognition and symbolic construction in terms of general arithmetic relations that are 'constructed' in the form of numbers or algebraic formulae.

According to the major commentators who have analyzed the passages in question, Kant is supposed to have drawn a strict distinction between symbolic and ostensive construction corresponding to a strict distinction between the methods of arithmetic and algebra, on the one hand, and geometry on the other. Such analyses interpret the algebraic symbols and operations thereon by analogy to numerical calculation and, moreover, suggest that 'symbolic construction' is either the inscription or manipulation of such symbols and the formulas in which they appear. The various interpretations employ these or analogous suppositions either to give a positive reading of the passages at issue, or to show that no such reading is possible. On my view, both such approaches are unsatisfactory: the former attributes to Kant an awkward and ultimately indefensible position, while the latter summarily dismisses any Kantian account of algebraic cognition.

Interpretations such as those I have been considering are weak in two related respects: they are poorly motivated with respect to both Kant's own text and the mathematics of his time. To illustrate, I will give two examples of this weakness. First, though Kant does not give arithmetic examples when he speaks of algebra and 'symbolic construction', the commentators nevertheless assume that these must be understood by analogy to the object and method of arithmetic, and *vice versa*, thus assimilating algebra and arithmetic in Kant's view. This assimilation is natural, given our twentieth-century understanding of elementary algebra. Kant himself, however, gives us no reason to suppose that he thought of the construction of arithmetic concepts as 'symbolic' or non-ostensive; on the contrary, Kant's arithmetic examples in the *Critique* appeal to ostensive constructions such as 'strokes' or 'points' (Kant, 1998, B15, A240/B299). Moreover, eighteenth-century methods of arithmetic and algebra, with which Kant was familiar and on which he based his conception of these mathematical sciences, are not isomorphic to their twenti-

[12]There are at least two additional and important problems with Friedman's view on this matter that are apparently unrelated to the assumptions made by Broad *et al.* First, he looks to a letter from Kant to Rehberg for confirmation that by algebra Kant meant the theory of incommensurable magnitudes (Friedman, 1992, p. 110). In fact, Kant does there write of the status of the concept of an irrational magnitude such as $\sqrt{2}$; however, he does not tie this concept, nor the rule for its approximation, to algebra. Neither does Kant claim that the rule for the approximation of an irrational magnitude constitutes a 'construction' of any kind. Rather, he calls for the ancillary necessity of a 'geometric construction of such quantities' (which would consist in drawing a line segment, such as the diagonal of a unit square) to adequately represent such quantities to the understanding (Zweig, 1967, pp. 166–9). Secondly, Friedman's suggestion that for Kant and his contemporaries algebra is roughly identifiable with the Eudoxean/Euclidean theory of proportion would seem to be belied by the mathematical texts from that time. 'Algebra' is typically included with trigonometry and calculus under the heading of 'Analysis', while the theory of proportion is given within the (independent) discussion of arithmetic. See, in particular, Wolff (1772, 1968, 1973).

eth-century counterparts.[13] In order to understand and appreciate Kant's view of ostensive and symbolic mathematical constructions we must first understand and appreciate the elementary disciplines of arithmetic, algebra, and geometry at their eighteenth-century stage of development.

Secondly, the suggestion that a 'symbolic construction' is a construction of or out of algebraic symbols is inconsistent with Kant's view of the role of construction in mathematical cognition. Recall that Kant invokes the construction of mathematical concepts to explain the syntheticity of mathematical judgments: the construction of the mathematical concept allows the mathematician to 'go beyond it [the concept] to properties that do not lie in this concept but still belong to it' thus forming a synthetic judgment (Kant, 1998, A718/B746). This power is unavailable to the philosopher, who reasons about concepts discursively without exhibiting them in intuition. Suppose, as the commentators have, that Kant meant the inscription of an algebraic formula to play a role in algebra akin to the role a geometrical figure plays in a geometric demonstration, namely to construct a mathematical concept in intuition and thereby reveal the properties of the object which falls under it. This supposition implies that the arbitrary marks chosen to express a mathematical relationship algebraically in the form of an equation are necessarily involved in the solution of that equation, just as the figures of the geometer are, for Kant, necessarily involved in the demonstration of a geometric theorem. Moreover, it implies that the single symbol chosen to 'construct' an algebraic concept in intuition, say 'x', is somehow able to reveal more about that concept than the philosopher's discursive and unconstructed concept, which Kant assures us is incapable of leading to synthetic mathematical judgments.

There is a crucial disanalogy, however, between constructing a geometric concept by producing a figure in intuition and constructing an algebraic concept by naming it with an arbitrary symbol; in the former case, the mathematician operates on the figure, paying attention to 'the act' whereby the concept is constructed in order to reveal and amplify the concept. In the latter case, the algebraic symbol is manipulated according to formal rules which are independent of the particular referent of the symbol; thus, an inscribed algebraic symbol, chosen arbitrarily to represent an algebraic concept, can in no sense provide an intuition that serves to reveal or amplify that concept. Consequently, the commentators' claim that 'symbolic constructions' of algebraic concepts are constructions of or out of algebraic symbols is inconsistent with Kant's view that the construction of mathematical concepts in intuition explains the syntheticity of mathematical judgments.

The shared assumptions I have identified in commentary on Kant's philosophy of algebra and symbolic construction are therefore suspect and must be rejected. Fortunately, attending to the details of the relationships among the elementary mathematical disciplines in the eighteenth century can lead us to a reading that is

[13]I am not supposing that Hintikka, Parsons, Friedman *et al*. believe that our modern notion of (abstract) algebra can be understood by analogy to elementary arithmetic. Rather, I am arguing against their assimilation of our modern notion of *elementary* algebra, i.e. as it is taught today in secondary schools, to the eighteenth-century notion of elementary algebra as Wolff and Kant understood it.

more satisfying, both textually and philosophically. The historical evidence will show that in the eighteenth century algebra was not considered a type of arithmetic, but rather a method for solving both arithmetic and geometric problems. Moreover, it will show that in applying algebra to the solution of such problems, algebraic symbols were used to symbolize lengths of constructed line segments and the relations among them; the algebraic symbolism thus provided a shorthand for manipulating geometrically constructible objects. Attention to this background of mathematical practice, which I will provide in the next section, facilitates a strong reading of Kant's philosophy of algebra which is historically accurate and well motivated by Kant's own text.

3.

During the thirty years preceding the publication of the second edition of the *Critique of Pure Reason*, Kant taught college-level courses in mathematics and physics. His mathematics courses were based on the popular, comprehensive textbooks by Christian Wolff: *Anfangs-Gründe aller Mathematischen Wissenschaften* (Wolff, 1973), *Auszug aus den Anfangs-Gründen aller Mathematischen Wissenschaften* (Wolff, 1772), and *Elementa Matheseos Universae* (Wolff, 1968).[14] The multi-volume *Anfangs-Gründe* covers the pure mathematical disciplines of arithmetic, geometry, trigonometry, and algebra, in addition to such 'mixed' mathematical disciplines as mechanics, optics, astronomy and artillery; a condensed version of the *Anfangs-Gründe* is presented in the single-volume *Auszug*. The *Elementa Matheseos Universae* gives a more thorough presentation of the pure mathematical disciplines and is divided as follows: the first half presents the elements of arithmetic, geometry, and trigonometry as well as an essay on the method of mathematics in general; the second half presents finite analysis, which includes 'specious arithmetic' and algebra, and infinite analysis, which includes differential and integral calculus. Not only did Kant use Wolff's textbooks in his courses, but, more generally, Wolff's texts were representative of the state of elementary mathematics when Kant was writing the *Critique*. Wolff's works therefore constitute an approved guide to Kant's understanding of the state of elementary mathematics. I will use them as a tool for clarifying the relationships between the objects and methods of arithmetic, geometry, and algebra in the eighteenth century, so as to provide context for Kant's remarks on mathematics.

First, it is important to realize that, for Wolff, algebra is not considered a science or discipline in its own right, but is rather an art or method which aids in the

[14]For more details on Kant's use of these texts, see the introduction to Martin (1985), particularly the translator's footnote 17 (Martin, 1985, pp. 143–4). While Martin's study of Kant's engagement with mathematical practice emphasizes the mathematical works published by Kant's students subsequent to the publication of his *Critique*, I am concerned to evaluate the texts that, preceding the publication of the *Critique*, would have directly influenced the conception of mathematics that is articulated therein.

solution of certain arithmetic or geometric problems (Wolff, 1968, pp. 297, 341).[15] Moreover, at the basis of Wolff's conception of the problems of arithmetic and geometry is his conception of number, magnitude, and unit. So, before examining the algebraic methods that Wolff employs in the solution of elementary mathematical problems, I will briefly discuss the object of such problems, that is, the mathematical ontology of Wolff's *Elementa*.

For Wolff, mathematics is the science of all that can be measured; arithmetic is the science of number, or counting; and a number is 'that which may be referred to unity' (Wolff, 1739, p. 1).[16] Insofar as numbers are employed to count, that which is measured by them according to the rules of arithmetic are particular, discrete objects. Any such object may be considered as the unit in relation to which all other objects will be conceived. For example, in his 'Elementa Arithmeticae' Wolff says that 'Two globes of stone are the same unities, but if one be of stone, another of lead, they are called different unities; but if they be considered only as globes, they are the same unities' (Wolff, 1739, p. 1). Thus, a number 'arises' upon considering some group of individual things of a particular kind with respect to a pre-selected unit (Wolff, 1973, p. 38). Numbers are homogeneous if they refer to the same unity, as 'two globes of silver' and 'three globes of silver'; they are heterogeneous if not, as 'two globes of silver' and 'three globes of lead' (Wolff, 1968, p. 26).

The preceding definitions characterize positive integers as wholes made up of discrete parts identical to an arbitrarily selected unit. Moreover, because straight lines are easily compared one to another, Wolff says that all numbers are best expressed by straight lines (Wolff, 1968, p. 24). Thus, once a single, given straight line is chosen as unity, any number can be constructed in relation to it, as the simple concatenation of units. More precisely, the line segment designated as the unit stands in the same ratio to the line segment for a given number as the number one stands to that number. Wolff goes on to define rational numbers as those commensurable with unity, and irrational as incommensurable; consequently, any straight line segment that is measurable by the chosen unit expresses a rational number. Thus, Wolff's understanding of numbers, the objects of arithmetic, ultimately relies on traditionally geometric concepts of line length, commensurability and incommensurability.[17]

Plane geometry is, for Wolff, the science of extended magnitude: it provides the rules for producing, measuring, and comparing straight-edge and compass constructible objects. The objects of plane geometry are both quantitatively comparable, as sized objects which bound equal or unequal areas or lengths, and qualitat-

[15]Since Euclid's *Elements* was considered to have produced all the theorems of plane Euclidean geometry, the science of geometry during the early modern period was conceived primarily as the art of solving geometrical problems (Bos, 1992, p. 23; Bos, 1993b, pp. 39–40). Algebra was conceived as a method for aiding in the solutions of such problems (Wolff, 1965, p. 34).

[16]Wolff (1739) is an eighteenth-century English translation of most of Wolff's *Elementa Matheseos*.

[17]Consequently, Wolff calls negative quantities 'absurd', 'privative of true', 'wanting reality', and 'not real' (Wolff, 1739, p. 72).

ively comparable, as shaped objects which are either similar or non-similar. The familiar theorems of elementary Euclidean geometry that Wolff rehearses in his *Elementa* demonstrate which properties hold of any and all objects constructed in accordance with the preliminary definitions, postulates, and axioms; the Euclidean problems show how to effect various constructions that are required in the demonstration of subsequent theorems.

For Wolff, the 'analytic art' or 'analysis' is a general method for solving certain kinds of mathematical problems.[18] Finite analysis finds 'from some known finite magnitude other finite magnitudes which are still unknown' (Wolff, 1965, p. 53). Algebra, as mentioned above, is a kind of finite analysis 'by means of equations' that is brought to bear on the solution of various types of arithmetic and geometric problems (Wolff, 1965, p. 34). Thus, for Wolff, algebra is a method of reasoning about the objects of elementary arithmetic and geometry, which include numbers (conceived as line segments) and plane geometric constructions. In addition to applying algebra to the solution of common determinate and indeterminate numeric problems, Wolff's *Elementa* is one among many eighteenth-century mathematics texts that also includes the then standard (and now obsolete) 'application of algebra to geometry'. This topic, the 'application of algebra to geometry', codifies the Cartesian programme for constructing equations, that is, for expressing geometric problems algebraically in equation form and then providing geometric constructions of the roots of those equations.[19]

At this point, we can see that the relationship between algebra, arithmetic, and geometry is not as the commentators discussed above in Section 2 have supposed: algebra was not simply a generalized arithmetic, but was rather a general method for the solution of particular problems of geometry and arithmetic. I will focus on the details of the 'application of algebra to geometry' in order to provide insight into how greatly the algebraic method with which Kant was familiar differs from our own.

In his textbook *Application de l'algebre à la geometrie*, the French mathematician Guisnée clearly states the purpose of such an enterprise:

[18]None of Wolff's terms such as 'analytic art', 'analysis', or 'application of algebra to geometry' can be taken to designate a strict translation of problems of geometry into problems of algebra, nor can they be associated with our notion of 'analytic geometry' *simpliciter*. For a discussion of these terms, and their relation to 'analytic' or 'coordinate' geometry, see Boyer (1956), especially Chapter 4.

[19]The method of constructing equations is discussed in all manner of mathematics texts throughout the seventeenth and eighteenth centuries: it is presented in geometry texts (Descartes, 1954; Lamy, 1758); in algebra texts (Harris, 1702; MacLaurin, 1748; Newton, 1967; Simpson, 1755); in general mathematics texts (Weidler, 1784; Wolff, 1968); and, in some cases, in texts devoted exclusively to the method itself (Guisnée, 1733). Boyer tells us that the title of Guisnée's text, *Application de l'algebre à la geometrie, ou Methode de démonstrer par l'algebre, les theorêmes de Geometrie, et d'en résoudre et construire tous les Problêmes*, contributed the name to this topical descendant of Descartes' *Geometrie* (Boyer, 1956, p. 149). Indeed, the section of Wolff's text which demonstrates the construction of equations is entitled 'Algebra ad Geometriam Elementarem Applicata'. For a lucid and interesting discussion of the history of the discipline of constructing equations, see Bos (1984) and his related articles Bos (1992, 1993a, b).

> On y explique le plus simplement que l'on peut, les methodes de démonstrer par l'Algebre, tous les Theorêmes de Geometrie, & de résoudre, & construire tous les Problêmes déterminez & indéterminez, geometriques & méchaniques. En un mot, on explique tous les usages qu'on peut faire de l'Algebre commune, dans toutes les parties des Mathematiques, pourvû qu'on exprime par des lignes les grandeurs qu'elles ont pour objet; & on ne suppose pour cela que les simples élémens de la Geometrie ordinaire. (Guisnée, 1733, p. ii)[20]

In order to fulfill these goals, the steps of Descartes' method for 'solving any problem in geometry' are explicitly invoked by Guisnée, Wolff and other participants in the codification of the Cartesian techniques. Note that Guisnée emphasizes the use of this method to resolve and 'construct' mathematical problems by first expressing any and all magnitudes geometrically, by straight line segments, and then reasoning about them algebraically. Algebra is brought to bear on problems about magnitudes that have been geometrically constructed; correspondingly, the solutions that algebra helps to find will also be in the form of geometrically constructible magnitudes.

Since we are primarily concerned with Wolff's *Elementa*, I will examine his particular presentation of this method for solving geometrical problems. Wolff begins his 'application of algebra to geometry' with three meta-mathematical problems: how to resolve geometric problems algebraically; how to construct simple equations; and how to construct quadratic equations (Wolff, 1968, pp. 385–388). The solution to these three meta-mathematical problems constitutes his method for solving subsequent particular problems of geometry by the application of algebra.

The solution to the first, how to resolve geometric problems algebraically, begins by invoking a previously solved problem: how to resolve *any* given problem algebraically (Wolff, 1968, p. 342). The first step toward solving a geometric problem algebraically is thus to follow the previously dictated steps required to prepare *any* problem for an algebraic solution. These steps can be summarized as follows: one first distinguishes the given or known magnitudes of the problem from those that are unknown by symbolizing the former by the first letters of the alphabet, and the latter by the last. Then, one finds as many equations as there are unknown magnitudes sought; the equations relate the known and unknown magnitudes according to the given conditions of the problem. Finally, the equations are transformed, or solved simultaneously, so that all unknown magnitudes are expressed algebraically in terms of known magnitudes.

The preparatory work having been completed, Wolff proceeds to the particular case of solving a *geometric* problem algebraically. Wolff here offers suggestions for finding the appropriate equations in geometrical problems, which he says is more difficult than in arithmetical problems. First, 'let us suppose the thing done,

[20]'One here explains in the simplest way one can, the methods of demonstrating by algebra all of the theorems of geometry, and the resolution and construction of all the determined and undetermined geometrical and mechanical problems. In a word, one explains all the uses that one can make of common algebra in all parts of mathematics, provided that one expresses the magnitudes that mathematics has for its object by lines; and one supposes for this only the simple elements of ordinary geometry.' This and all subsequent unattributed translations are my own.

which is proposed to be done' (Wolff, 1968, p. 385). This is a terse prescription to draw a figure, or scheme, that represents the construction required by the geometric problem at hand; all elements of this figure that are characterized by the conditions of the problem will be labelled according to whether exhibited magnitudes are known or unknown.

To see what this suggestion amounts to, consider the first problem to which Wolff applies his method: 'Given the perimeter AB + BC + CA and the area of a right-angled triangle, to find the hypotenuse.' (Wolff, 1968, p. 388) In this problem (which I will call the Hypotenuse Problem) the perimeter and the area are given as single one-dimensional magnitudes, the lengths of two constructed straight line segments.[21] Both the perimeter and the area are constructed relative to a given unit, also a constructed straight line segment (Fig. 1); being known magnitudes, they are both symbolized by letters from the start of the alphabet, a and b^2 respectively.[22] Since the hypotenuse is sought and not given, its length is unknown and its construction is called for in the solution of the problem; thus, the triangle itself cannot be constructed according to the scale of the given magnitudes until the length of the hypotenuse has been found.[23] Nevertheless, a schematic triangle can be drawn immediately, one not constructed according to the scale of the given magnitudes, but rather a simple representation of any right triangle. On this sche-

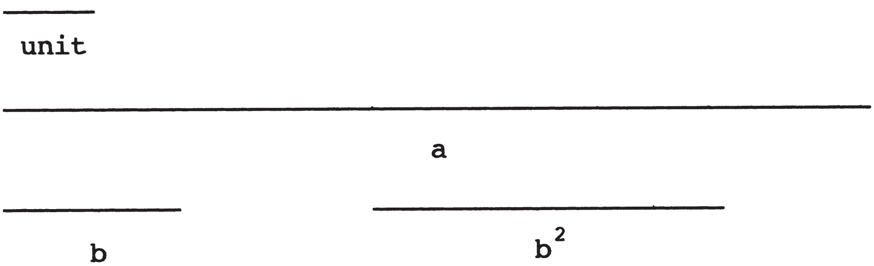

Fig. 1.

[21]In addition to the perimeter, Wolff actually gives a straight line segment equal to the square root of the area; the area can then be constructed as a straight line segment by multiplying the root by itself one-dimensionally, according to Descartes's procedure for multiplying line segments, thus 'squaring' it (Descartes, 1954, p. 5). Alternately, the area could have been given as a square constructed on a given straight line segment (equal to the square root of the known area). If so, the area would be transformed into a line segment by the Cartesian procedure given in Rule 18 of the *Rules for the Direction of the Mind* (Descartes, 1985, p. 76).

[22]Explaining his choice of b^2 to symbolize the area, Wolff explains in a scholium: 'Seeing we measure the areas of figures in geometry, by finding out the ratio they have to some given square, they are likewise found in Algebra by the side of some square that is equal to them.' (Wolff, 1739, p. 174)

[23]As it turns out, the construction of the hypotenuse is necessary but insufficient for the construction of the whole triangle. Wolff goes beyond the stated requirements of the problem and, finding the altitude in addition to the hypotenuse, is able to construct the triangle itself according to the conditions specified. For the purposes of the exegesis of Wolff's general method, I will show only how to construct the hypotenuse itself, as a single straight line segment.

matic diagram the vertices are labeled A, B, and C; the hypotenuse labeled x; the altitude meets AC at D and is labeled y (Fig. 2).

Returning to Wolff's suggestions for solving *any* geometric problem algebraically: he next says that the lines of the 'scheme' so drawn must be examined (and possibly produced or joined) in order to determine the relations that hold between the known and unknown magnitudes of the problem. He advocates the Cartesian principle that the figure should be drawn so as to facilitate the use of two Euclidean theorems in particular: the proportionality of similar triangles, and the Pythagorean theorem.[24] Indeed, he remarks that seldom are any theorems but these invoked to express the geometric relationships between the magnitudes (Wolff, 1968, p. 385). The necessary algebraic equations result when the relationships made palpable by these geometric theorems are expressed symbolically. At this point, the equations can be transformed and reduced so that the unknown magnitude is expressed as simply as possible in terms of relations between known magnitudes.

In the case of the Hypotenuse Problem begun above, recall that the perimeter and area were given as straight line lengths, symbolized by a and b^2, respectively. Also, we have a schematic triangle with the hypotenuse AC labeled x, the unknown magnitude which is to be 'found'. Wolff arrives finally at an equation for the unknown magnitude, the hypotenuse:[25]

$$x = 1/2 \cdot a - 2b^2/a. \qquad (*)$$

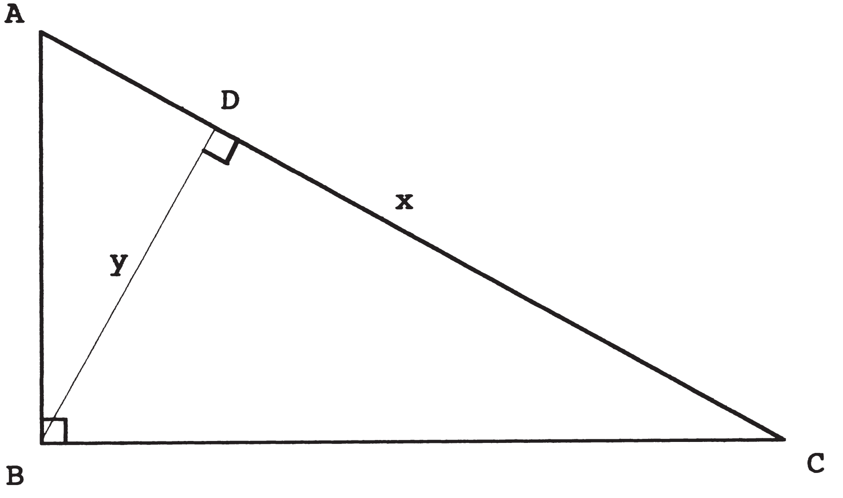

Fig. 2.

[24]Descartes states in a letter 'In the solution of a geometrical problem I take care, as far as possible, to use as lines of reference parallel lines or lines at right angles; and I use no theorems except those which assert that the sides of similar triangles are proportional, and that in a right triangle the square of the hypotenuse is equal to the sum of the squares of the sides. I do not hesitate to introduce several unknown quantities, so as to reduce the question to such terms that it shall depend only on these two theorems.' (Descartes, 1954, p. 10, n. 18)

[25]For Wolff's derivation of this equation, see the Appendix.

He now has an expression for the unknown magnitude *x*, the length of the hypotenuse AC, in terms of the known magnitudes *a* and b^2.

Again returning to Wolff's instructions, we come to the final step of his procedure for solving a geometric problem algebraically: 'Having reduced the equations, you may from the last make a geometrical construction of it several ways, according to the difference of equations.' (Wolff, 1968, p. 386) At this point in the problem, one has arrived at a symbolic expression of the unknown magnitude in terms of known magnitudes. If the problem were numerical, that is, if the known magnitudes were not known only as constants (i.e., symbolized by letters from the start of the alphabet), but known by their actual numerical values, then the solution to the problem would be computed by substituting numerical values for the constant terms of the equation. But we see that in the case of a geometrical problem a symbolic expression or numerical value does not suffice as a solution, even to a geometrical problem that is solved algebraically. Rather, Wolff's procedure calls for a step beyond the reduction of equations: the final solution of a geometrical problem solved algebraically amounts to the *geometrical construction of the magnitude represented by the symbol for the unknown quantity sought for in the problem*. In the problem at hand, the expression for *x* given by Equation * is not a sufficient solution: because all magnitudes are conceived as straight line segments, 'to find' the hypotenuse means to construct it geometrically, as a straight line segment.

For Wolff, then, the symbolic expression of a geometric magnitude is insufficient as a final solution of a geometric problem; just as the geometric schemes or figures were constructed to enable cognition of the relationships between the symbolically expressed magnitudes, so must a geometric figure be constructed to satisfactorily *show* the referent of the symbol for the sought-for unknown magnitude. At this point in the problem, the equations have been reduced so that the symbol for the unknown magnitude stands alone on one side of an equation, while the other side expresses it in terms of the arithmetic combination of symbols for known magnitudes only; now, the geometric construction of the former is effected by a geometric construction of the latter. Such a construction is, appropriately, called the 'construction of an equation'.[26]

Accordingly, Wolff's next two meta-mathematical problems explain how to construct geometrically the standard forms of simple and quadratic equations; that is, how to construct geometrically the *roots* of the fully reduced equations in any problem, thereby exhibiting the unknown magnitude(s) sought for. His construction of a simple degree-one equation proceeds by algebraically manipulating the equation into a proportion such that the symbol for the unknown magnitude stands

[26]Thus, in his *Lexicon* Wolff gives the following definition: '*Constructio aequationum, effectio geometrica, die Ausführung der Gleichungen*, Wird genennet, wenn man durch hülffe Geometrischer Figuren den Werth der unbekandten Grösse in einer Gleichung in einer geraden Linie findet; oder sie ist die Erfindung einer geraden Linie, welche die unbekandte Grösse in einer Algebraischen Gleichung andeutet.' (Wolff, 1965, p. 421) 'It is called *the construction of equations, effected geometrically*, when with the aid of geometrical figures the value of the unknown magnitude in an equation is found in a straight line, or it is the finding of a straight line which the unknown magnitude in an algebraic equation indicates.'

in the place of the fourth proportional, for example a:b::c:x; then, the geometric construction is easily effected in the manner of Euclid's VI.12.[27] More complicated equations, including quadratics, require techniques that reduce them to simple equations; once reduced, they can be constructed in like fashion.[28]

In order to construct the root of Equation *, namely x, and thereby 'find' the hypotenuse, Wolff proceeds as follows. He first sets out the straight line BD = a, raises the perpendicular BA = $2b$ from B, and lets BG = b (Fig. 3).[29] He then constructs the fourth proportional to a, $2b$, and b by first connecting D and A, and then drawing a line parallel to DA from G, meeting BA at H. By similar triangles, a:$2b$::b:BH, and, thus, BH = $2b^2/a$. This expression looks familiar; indeed, it is the second term in the algebraic expression for the hypotenuse x (see Equation * above). Thus, Wolff has chosen constructions of the given magnitudes that facilitate construction of the components of the equation for the desired hypotenuse.

Next (Fig. 3), letting BC = $1/2 \cdot a$ (by simple bisection of BD, as warranted by Euclid I.10), and CI = BH (by Euclid I.2), it follows that since

$$\mathrm{BI} = \mathrm{BC} - \mathrm{CI},$$

then, by substitution,

$$\mathrm{BI} = 1/2 \cdot a - 2b^2/a.$$

By Equation *, BI = x. Thus, Wolff has effected the construction of the hypotenuse

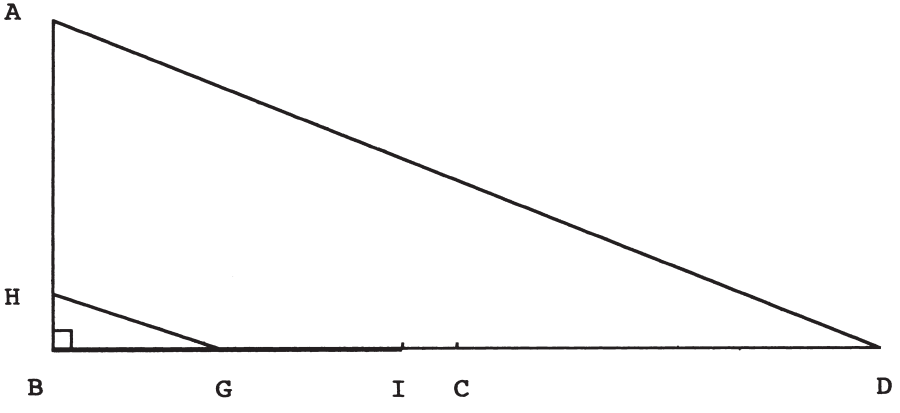

Fig. 3.

[27]Wolff is assuming the standard constructions of simple arithmetic combinations of magnitudes; e.g., the sum of two known magnitudes is constructed by a concatenation of the line segments that represent them, etc. See Descartes (1954), pp. 2–5.

[28]Because Wolff takes negative magnitudes to be 'unreal' (see [17], above), he does not give the constructions of negative (or imaginary) roots of equations, calling them 'false'. What he considers a 'true' solution is that which is geometrically constructible, and 'false' otherwise.

[29]Remember that Wolff takes the segment b to be given; Euclid's I.2 and I.3 then enable him to construct the given straight lines b and a from any endpoints, and to lay off the lesser on the greater. $2b$ can be constructed from the endpoint B by raising a perpendicular to BD at B (by Euclid I.11), concatenating two segments equal to b, and laying the latter off on the former.

of the desired triangle, x, at BI by *constructing the equation for x*, first constructing the second term, $2b^2/a$, and then subtracting it from the first, $1/2{\cdot}a$.[30]

We see clearly that Wolff does not consider Equation * to constitute a solution to the problem of finding the hypotenuse of the triangle. Rather, the solution to the problem, in accordance with Wolff's instructions, is the geometrically constructed root of the equation for x. Just as the letters a and b symbolize the known magnitudes that are given as straight line segments, the symbolic value of x, namely $1/2{\cdot}a - 2b^2/a$, symbolizes the true solution of the problem: the sought-for (and found) magnitude constructed as a straight line segment, as shown by BI (Fig. 3).

For Wolff, both known and unknown magnitudes are represented by straight line segments for the purposes of arithmetic and geometric problem solving. Algebra is a method which aids in the solution of such problems by symbolically manipulating such magnitudes: particular letters symbolize the known and unknown magnitudes of a particular problem,[31] and algebraically expressed operations on these symbols symbolize arithmetic operations on the magnitudes themselves.[32] Moreover, all such magnitudes, including numeric quantities, are themselves always understood in relation to an arbitrary, preselected unit; such a unit is arbitrary insofar as the choice of which particular magnitude will act as unit varies from problem to problem. In the case of geometrical problems, the unit is given as the length of a constructed straight line segment in relation to which all other magnitudes of the problem will be conceived; in arithmetical problems, the unit is a particular kind of object to which all other countable objects are or are not homogeneous.

Thus, for Wolff, algebra is not strictly identifiable with a general arithmetic of variable or indeterminate numeric quantities; the use of algebra to solve indeterminate arithmetic problems constitutes only one aspect of its applicability for Wolff. Moreover, the process of applying algebra to the solution of both arithmetic and geometric problems does not terminate with a 'symbolic' solution in either case. When the problem is arithmetic, the solution is typically numeric, and numerical solutions are computed by substituting known numerical values for symbolic expressions. When the problem is geometric, the solution is provided by an actually

[30]Wolff goes on to complete the construction of the triangle itself, an elaborate process which requires that he first find an equation for and construction of the altitude of the triangle. It is illuminating to compare such a final construction of the 'solution' triangle to the initially drawn 'schematic' triangle, which represented it for the purposes of labelling the elements of the problem: in the case of the 'solution' triangle, the constructed figure is not simply representative of the geometric relations between the elements of the problem. Rather it is an actual scale construction of the hypotenuse and triangle sought relative to the arbitrarily chosen unit of measure.

[31]E.g., $a, b, c, \ldots$ and $x, y, z, \ldots$ respectively.

[32]E.g., $a + b$ symbolizes the concatenation of line segments called a and b; $a - b$ symbolizes the diminishing of the line segment called a by the line segment called b; $a{\cdot}b$ symbolizes the construction of the fourth proportional c such that unit:a::b:c; $a \div b$ symbolizes the construction of the fourth proportional c such that b:a::unit:c; a^2 symbolizes the construction of the fourth proportional c such that unit:a::a:c; and, finally, $\sqrt{a}$ symbolizes the construction of the mean proportional c such that unit:c::c:a. Compare Descartes (1954) pp. 2–6.

constructed geometric object.[33] The point is, simply, that for Wolff 'algebra' does not demarcate its own class of objects or constructions ontologically distinct from the discrete and continuous magnitudes of arithmetic and geometry, but rather provides a method of reasoning about the constructible objects of both.[34]

We are left with an obvious question: *why* does Wolff consider a symbolic solution to a geometrical problem insufficient, even when the method of solution is algebraic? In answering this question, I will conclude my discussion of Wolff and make some initial suggestions as to how Kant's philosophy is illustrated by Wolff's example.

The answer to this question lies in the distinction between the theorems and problems of elementary mathematics, especially geometry. For early modern mathematicians, Euclid's *Elements* represented a complete set of the theorems of elementary geometry; what still required the geometer's attention were new or previously unsolved problems. The Hypotenuse Problem, for example, is formulated on the basis of particular given data and solved by using that data to produce a particular triangle of a particular size; the perimeter and the area are given by their actual extensive magnitudes, as straight line segments, and are symbolized in order to see the relations that obtain between the elements of the problem more easily and more clearly. The symbolization of the magnitudes is a heuristic step toward the solution of this particular problem; the symbolization is neither part of the given data of the problem, nor an attempt to strip the problem of its particular given data.

To see this more clearly, notice that the problem is *not* formulated in general, as follows: 'Given a right triangle with perimeter a and area b^2, find the hypotenuse.' Were this the original problem, the solution would surely be given as an equation in terms of a and b^2; such an equation would provide a general rule for finding the hypotenuse of *any* right triangle whose perimeter and area are known. But Wolff's problem concerns a particular constructible triangle, one with a particular constructed perimeter and a particular constructed area. Accordingly, the problem is solved on the basis of the particular given parameters: the unique particular hypotenuse that completes the triangle with such a given perimeter and area must itself be constructed. Admittedly, the problem provides a general rule for finding the hypotenuse of *any* right triangle with known perimeter and area; nevertheless, such a rule is inferred by analogy to the solution for this particular triangle and is not given in the general form of an equation. Rather, the 'rule' shows how to geometrically construct any such hypotenuse from the line segments given as perimeter and area.

[33]Lamy explains why the solution to a geometric problem cannot be numeric: 'Il faut observer d'ailleurs que les lettres qui représentent les lignes dans la solution d'un problême géométrique, ne pourroient pas toujours être remplacées par des nombres. Telle seroit la diagonale d'un quarré, indiquée par *b*, & dont le côté seroit *a*. Ainsi en assignant une valeur numérique à ce côté *a*, il seroit impossible d'assigner une valeur numérique à *b*, parce qu'elle est incommensurable. Mais en construisant le quarré sur la ligne *a* & tirant la diagonale, on aura la valeur de *b*.' (Lamy, 1758, p. 520)

[34]Numbers, as objects of arithmetic, are constructible for Wolff since he conceives them as ratios of straight line segments (Wolff, 1965, p. 944; 1968, p. 24).

To recast the point using Kant's terminology, such a rule is inferred by attending not to the particular characteristics of the constructed hypotenuse BI (Fig. 3), but rather to the 'act' whereby the hypotenuse was constructed, that is, to the series of geometrical constructions out of which BI emerged as the hypotenuse of the right triangle with given perimeter and area. It is in this sense that, for Kant, the particular individual object, in this case the hypotenuse BI, can exhibit the concept and express 'universal validity for all possible intuitions that belong under the same concept', namely, the concept of the hypotenuse of a right triangle with given perimeter and area (Kant, 1998, A713/B741).

4.

I am now in a position to show how an understanding of the eighteenth century conception of the algebraic method can provide insight into Kant's theory of algebraic cognition and 'symbolic construction'. Since Kant is known to have studied and taught Wolff's mathematics texts for many years, we can assume that his conception of mathematics was influenced by their presentation, which was comprehensive and generally representative of eighteenth-century college-level mathematics. Accordingly, by bringing to bear a picture of eighteenth-century mathematical practice (particularly algebraic practice as exemplified in Section 3), I will offer a new reading of the two relevant passages from the *Critique*.

To begin, we must clarify some of the terms Kant uses when he philosophizes about the method of mathematics. First, Kant uses the term 'magnitude' ('Größe') in two senses: he uses 'magnitude' to designate an object which has a particular determinate size, while also speaking of a thing's 'magnitude', that is, the size of an object. In other words, for Kant, magnitudes *have* magnitude.[35] Magnitudes in the first sense, as sized objects, are the constructible objects of geometry; these are conceived both quantitatively and qualitatively, that is, with respect to both their size and their shape, or figure. Magnitude in the second sense, as the size *of* an object, is to conceive magnitude in the first sense with respect to its quantitative aspect only; that is, to conceive of size without shape, quantity without quality. Used in this second sense, the term 'Größe' also refers to the pure concept of quantity, or the application of the category 'quantity' to a particular sized object.

Kant expresses this distinction in the 'Axioms of Intuition' where he states:

> Now the consciousness of the homogeneous manifold in intuition in general, insofar as through it the representation of an object first becomes possible, is the concept of a magnitude (*Quanti*) . . . appearances are all magnitudes, and indeed *extensive magnitudes*, since as intuitions in space or time they must be represented through the same synthesis as that through which space and time in general are determined. (Kant, 1998, A162/B203)

Then, Kant claims that such magnitudes (*quanta*) are the concern of the axioms

[35]In Latin the distinction is between 'quantum' or 'quanta' and 'quantitas'.

of Euclidean geometry. Kant contrasts this sense of magnitude to *quantitas*: the answer to the question of how big a thing is.

Appearances are, as extensive magnitudes, 'intuited as aggregates (multitudes of antecedently given parts)' (Kant, 1998, A163/B204). Thus, when the magnitude (*quantitas*) of a magnitude (*quantum*) is to be determined, one asks how many of such antecedently given parts make up the whole; the answer to such a question is expressed numerically by considering each of the discrete parts as homogeneous units. For a thing to be a magnitude (*quantum*) and, thus, to have magnitude (*quantitas*) is for it to be equal to a number of others taken together (Kant, 1998, A235/B288).

The determination of the magnitude (*quantitas*) of a magnitude (*quantum*), that is, finding the answer to the question of how many of some antecedently given parts make up a particular object, is what Kant designates as the 'magnitude in general' of a thing. He states that

> No one can define the concept of magnitude in general [*Größe überhaupt*] except by something like this: That it is the determination of a thing through which it can be thought how many units are posited in it. (Kant, 1998, A242/B300)

When an object is thought under Kant's concept of 'magnitude in general' it is quantified as an aggregate of units, and thus measured, or counted; this explanation should recall Wolff's definition of number (Section 3).[36] Thus, the magnitude (*quantitas*, size) of a magnitude (*quantum*, sized object) is determined by considering the object under the concept of 'magnitude in general', that is, by quantifying it relative to a chosen unit.

I am now prepared to tackle the first passage cited above in Section 1. Kant states: 'But mathematics does not merely construct magnitudes (*quanta*), as in geometry, but also mere magnitude (*quantitatem*), as in algebra, where it entirely abstracts from the constitution of the object that is to be thought in accordance with such a concept of magnitude.' (Kant, 1998, A717/B745) Here, Kant is reasserting his familiar claim that the objects of geometry are constructed by the geometer in the usual ways; these constructed objects (such as lines, triangles, and circles) are examples of 'magnitudes', that is, sized objects. He adds that the algebraist 'constructs magnitude' in the second sense discussed above: the algebraist ignores the qualitative aspect of the object, considering it only in accordance with the pure concept, or category, of magnitude (*quantitatem*). Thus, according to Kant, the algebraist can construct a mathematical object *qua* quantity only, without regard for shape or figure.

If, as a geometer, I construct a triangle, then I thereby construct a *quantum*, a sized object; but what does it mean for the algebraist to construct the *size* of an

[36]Kant's concept of 'magnitude in general' is not to be confused with the concept of 'number in general' which he invokes in the 'Schematism'. A reading of the 'Schematism' would be required to fully delineate these concepts, a task I will pursue elsewhere. For my purposes in this paper, it is sufficient to emphasize that 'magnitude in general' cannot be identified with a concept of a variable numeric quantity.

object without constructing the object itself? Such a construction of 'mere magnitude' would, presumably, represent the answer to the question: how big? That is, the algebraist's construction must exhibit how many of some antecedently given homogeneous units make up the particular sized object in abstraction from the construction of the object itself. But what sort of construction can do that, and how?

My examination of Wolff's application of algebra to the solution of mathematical problems in general (Section 3) will help to answer this question. Recall that Wolff considers any magnitude to be expressible by the length of a straight line; moreover, in the context of any particular mathematical problem, a unit magnitude is given or selected in terms of which all known magnitudes are constructed. Finally, the unknown magnitudes are constructed in terms of the unit and the known magnitudes. These constructions are effected in the Cartesian tradition by virtue of geometric interpretations of arithmetic operations; in particular, the operation of multiplication relies on the geometric construction of the fourth proportional.

We have seen that Kant's basic understanding of magnitude (*quantitas*) mirrors Wolff's: they both consider our determination of magnitude to depend on the number of times some particular unit occurs as part in the whole sized object. Moreover, Kant follows Wolff in taking the fourth proportional to three known magnitudes to be constructible; in the 'Analogies of Experience' he writes:

> In philosophy analogies [*Analogien*] signify something very different from what they represent in mathematics. In the latter they are formulas that assert the identity of two relations of magnitude [*Größenverhältnisse*], and are always *constitutive*, so that if two members of the proportion are given the third[37] is also thereby given, i.e., can be constructed. In philosophy, however, analogy is not the identity of two *quantitative* but of two *qualitative* relations, where from three given members I can cognize and give *a priori* only the *relation* to a fourth member but not *this* fourth *member* itself . . . (Kant, 1998, A179–180/B222–223)

It is important to note that the term 'analogy', when used in a mathematical context, is synonymous with 'proportion', both of which Wolff defines as 'die Aehnlichkeit zweyer Verhältnisse' (Wolff, 1965, p. 1105). Since the German 'Verhältnisse' is, in a mathematical context, the term for 'ratio', Kant is claiming that a mathematical analogy or proportion is a formula equating two ratios such that two (or three) given terms provide for or 'constitute' the construction of the third (or fourth).[38] Thus, Kant is aware that the mathematical proportion is not only able to specify the relations or ratios between the known and unknown magnitudes, but also provides for the construction of the unknown magnitude (the fourth proportional) *itself*.

[37]Kant's text reads, in part, '. . . so, daß, wenn zwei Glieder der Proportion gegeben sind, auch das dritte dadurch gegeben wird . . .' (Kant, 1990, A179/B222) Following Mellin, Kemp-Smith reads 'drei' for 'zwei' and 'vierte' for 'dritte'; Guyer and Wood read 'zwei' and 'dritte'. Either way, the text is plausible. If two members of a proportion are given, then the third can be found in continued proportion: for example, given a and b, x can be found such that a:b::b:x. More commonly, if three members of a proportion are given, then the fourth can be found: a:b::c:x. Further along in the passage Kant speaks of 'three given members' and 'a fourth member', in which case he clearly means a case such as a:b::c:x; in the mathematical case, the 'fourth member itself' is constructible.

[38]See previous note.

Now, returning to the passage at A717/B745, what Kant means by the construction of 'mere magnitude' is the construction of a straight line segment the length of which represents the size of a particular object, without respect to its quality, or shape. Such a straight line segment is, as we have seen, constructed in accordance with the quantitative relation, or proportion, that holds between the magnitude (*quantitas*) being constructed and the size of some known magnitudes, including the unit. Thus, the size, or 'mere magnitude', of any magnitude (*quantum*) can be constructed as a straight line segment, whatever the quality, shape or 'constitution' of the original *quantum*; to so construct the 'mere magnitude' of a thing is to consider it in accordance with the pure concept of quantity and to construct its 'magnitude in general'.

For example, in the Hypotenuse Problem (Section 3), the area of the triangle can be constructed either as a triangular figure (upon completion of the problem), or as Wolff exhibits it, as a straight line segment. In the former case, the 'object itself' is constructed; in the latter, the 'mere magnitude' of the area is constructed in abstraction from the 'constitution' or shape of the triangular object.

The algebraist is uniquely qualified to symbolize such a construction of 'mere magnitude': recall the eighteenth-century technique of constructing equations.[39] Descartes, Wolff and others consistently remark that the letters of algebra symbolize the straight line segments that represent the known and unknown magnitudes of any particular problem; moreover, the acceptable solutions depend on a geometrical construction of the unknown magnitude upon finding its proper algebraic expression. Thus, in the first passage Kant goes on to say that a 'notation [*Bezeichnung*]' is chosen for all 'construction of magnitudes in general (numbers), as well as addition, subtraction, extraction of roots, etc . . .': the construction of 'magnitudes in general [*Größen überhaupt*]' proceeds in the Wolffian manner as the construction of straight line segments relative to the chosen unit; arithmetic operations thereon are given the usual geometric interpretation; and letters are chosen to designate the various magnitudes of the problem, with compound algebraic expressions for the operations between them formed in accordance with the rules of 'specious arithmetic'.

Further, this notation, or lettering, is capable of symbolizing not only each particular magnitude of a problem, but also all possible constructions thereon; so, once the relations between all of the magnitudes of a problem are specified, that is, once ratios or proportions are constructed between the known and unknown magnitudes according to the given conditions of the problem, the algebraist can symbolize each such relation by writing equations using the chosen notation.[40] Thus, when Kant

[39]We know that Kant was familiar with the method of constructing equations. In what appears to be a draft of a 1790 letter to Rehberg (Zweig, 1967, pp. 166–9), Kant 'admires' the fact that the (non-contradictory) relations among magnitudes thought *arbitrarily* by the understanding always find corresponding intuitions in space; to even irrational concepts of magnitude there thus corresponds an 'object'. This Kant attributes to the construction of equations: 'Daher auch der Anfänger (in der Algebra) bey der (geometrischen) Construction der Aequationen durch das Gelingen derselben mit einer angenehmen Bewunderung überrascht wird.' (Kant, 1911, p. 58)

[40]Compare Wolff's procedure for deriving Equation *, as outlined in the Appendix.

speaks of the 'procedure through which magnitude is generated and altered in accordance with certain general rules in intuition' he refers to the geometrical constructions whereby the known and unknown magnitudes are arithmetically manipulated: for example, when two magnitudes are added together by concatenating line segments; a root is extracted by construction of the mean proportional; or an unknown quantity is constructed as a fourth proportional. When Kant speaks of the algebraist 'exhibiting' these procedures, or relations, he means that the algebraist writes a symbolic expression that stands for a particular geometric construction: such as $a + b$; $\sqrt{a}$; a:b::c:x; or $x = (bc)/a$.

Finally, referring to an algebraic expression such as $a \div b$, Kant says:

> . . . where one magnitude is to be divided by another, it [mathematics, or algebra] places their symbols together in accordance with the form of notation for division, and thereby achieves by a symbolic construction equally well what geometry does by an ostensive or geometrical construction (of the objects themselves) . . . (Kant, 1998, A717/B745)

The algebraic expression $a \div b$ symbolizes the construction of the quotient of the magnitudes symbolized by a and b; that is, $a \div b$ symbolizes the geometrical construction of the fourth proportional x such that b:a::unit:x. The algebraic expression is short-hand for the sometimes tedious geometric construction of 'the objects themselves', in this case the straight line segments a, b, and x.

The insight Kant expresses here is that the symbolic expression, though inextricably tied to its geometrically constructible referent, can nevertheless stand alone for the problem-solving purposes of the algebraist. Kant recognizes that, for problem-solving purposes, the algebraist 'achieves' equally well *via* an expression like $a \div b$ what the geometer does *via* a construction of the fourth proportional x: in both cases, the quotient is determined. This does not mean, however, that Kant takes either the expression $a \div b$ or the letters and operation symbols it comprises to be the constructed object of mathematics; the 'symbolic' construction $a \div b$ does not achieve the construction of the mathematical concept in intuition, but rather provides a clear and easily manipulable symbol of the performable construction of the fourth proportional to b, a, and the unit. Whether or not the construction is actually carried out, or merely symbolized, the ostensive straight line magnitudes or 'objects themselves' are that which, for Kant, exhibit the concept in intuition.

At A734/B762 Kant explains further:

> Even the way algebraists proceed with their equations, from which by means of reduction they bring forth the truth together with the proof, is not a geometrical construction, but it is still a characteristic construction, in which one displays by signs in intuition the concepts, especially of relations of quantities, and, without even regarding the heuristic, secures all inferences against mistakes by placing each of them before one's eyes. (Kant, 1998, A734/B762)

Following the same example, the equation $x = a \div b$ (having been reduced to its simplest form such that the unknown magnitude x is expressed as the root of a one-degree equation in terms of known magnitudes a and b) is a 'characteristic construction'[41] displaying the concepts of the magnitudes *a*, *b*, and *x*, and the relations thereon, without effecting a full geometric construction of the objects themselves. Rather, the equation for *x* symbolizes that magnitude *via* the 'characters' or symbols for the mathematical elements and relations which constitute it (i.e., the known magnitudes *a* and *b*, and $\div$, the procedure for dividing magnitudes by construction of a fourth proportional).[42]

Kant closes with a simple comment on the advantages of such a symbolic, algebraic method, a comment which recalls Descartes' own insistence that algebraic symbolism keeps the mathematician focused on the crucial elements of a problem, and is a useful aid to memory (Descartes, 1985, pp. 66–69). Kant also recognizes that such a method of solving geometrical problems has the advantage of 'bring[-ing] forth the truth together with the proof'; that is, the derivation or 'reduction' of equations plays the dual role of leading to the equation whose root is the solution sought (i.e., the 'truth') while also constituting a proof of the validity of that solution by deriving it step-by-step according to prescribed rules.[43]

I foresee a possible objection to my reading. From this last passage, we might interpret Kant as taking the algebraic expression itself to be a 'construction' of the *relation* between magnitudes, as opposed to a construction of the magnitudes themselves, or a symbol of such. In this case, the inscription of, for example, the equation $x = a \div b$ would itself provide a construction of the quantitative relation, or proportion, b:a::unit:x. To clarify: I have argued against any interpretation that implies that the inscription $a \div b$ is itself a constructed object, in favor of an interpretation such that $a \div b$ symbolizes the construction of the fourth proportional *x*. I now wish also to reject the possible suggestion that the equation $x = a \div b$ is a construction of the *relation* that holds between the three magnitudes, *x*, *a*, and *b*.

Kant makes this point clearly in a letter to Reinhold in which he addresses Eberhard's ignorance of mathematics and misunderstanding of his *Critique*. There, Kant writes:

> The mathematician can not make the least claim in regard to any object whatsoever without exhibiting it in intuition (or, if we are dealing merely with quantities without

[41]Guyer and Wood note that: 'Here Kant is using "characteristic" (*characteristisch*) in the sense of a computational method in which concepts are assigned numerical values, the sense underlying Leibniz's project of a "universal characteristic", in which all questions could be solved by analysis by assigning a numerical value to all concepts.' (Kant, 1998, A734/B762, n. 12) My interpretation of the passage is unaffected by whether the concepts of magnitude are assigned actual numerical values, or are simply symbolized by letters. The German 'charakteristische Konstruktion' could also be read in the sense of a 'typical' construction, in which case Kant is simply saying that the algebraist effects a typical construction symbolically.

[42]This point is made still clearer if we translate the portion of the passage that reads '. . . in welcher man an den Zeichen die Begriffe, vornehmlich von dem Verhältnisse der Größen, in der Anschauung darlegt . . .' as follows: '. . . in which one displays the concepts in intuition in signs, especially the concepts of the relations of quantities . . .' The actually constructed geometric object (e.g., the fourth proportional) is a 'concept in intuition' which, in the case of a 'characteristic construction', is represented by a sign or symbol that stands for it. I am grateful to Lanier Anderson for bringing this point to my attention.

[43]For an example of such a derivation, see the Appendix.

> qualities, as in algebra, exhibiting the quantitative relationships thought under the chosen symbols). (Allison, 1973, p. 167)[44]

The algebraic expression, or equation, such as $x = a \div b$, 'stands for' the quantitative relationship between x, a, and b such that b:unit::a:x. For this quantitative relationship to be 'exhibited in intuition' must be for the object x to be constructed geometrically as a line segment relative to the line segments a and b and in accordance with the usual construction of the fourth proportional.[45] Thus, the expression $x = a \div b$ is not itself a construction of an abstract relation; rather, it *symbolizes* the geometrical construction of x in relation to a and b.

Kant's technical term 'construction', as in the 'mathematical construction of concepts', is used throughout the *Critique* to designate the production of a geometrical figure which is characterized by its ability to be ostended or exhibited in intuition.[46] Even specifically arithmetic concepts are, like geometric ones, ostensively constructible as figures for Kant; at B15 he cites Segner's *Anfangsgründe der Arithmetik* in which numbers are constructed as both line segments, in the same manner as Wolff, and arrangements of dots (Segner, 1773, pp. 4–5, p. 27, p. 79). Later, at A240/B299, he refers to the 'strokes and points' used, especially by Segner, to construct numeric concepts.[47] When Kant says at A717/B745 that algebra 'achieves by a symbolic construction equally well what geometry does by an ostensive or geometrical construction (of the objects themselves)' he does not mean to draw a strict distinction between symbolic construction, on the one hand, and ostensive/geometrical construction on the other. Insofar as algebra is a method applied to the solution of mathematical problems, the algebraic expression symbolizes the construction of arithmetic and geometric concepts in the form of figures. Thus, 'symbolic constructions' are not *kinds* of constructions, that is, constructions of or out of symbols or characters. Rather, they are that which symbolize ostensive, or geometrical constructions.

Kant makes this point clear in his essay 'On a Discovery According to which Any New Critique of Pure Reason Has Been Made Superfluous by an Earlier One', in which he responds to Eberhard's criticisms of his *Critique*. The passage, which

[44]'Denn es ist gerade umgekehrt: sie können nicht den mindesten Ausspruch über irgend einen Gegenstand tun, ohne ihn (oder, wenn es bloß um Größen ohne Qualität, wie in der Algebra, zu tun ist, die unter angenommenen Zeichen gedachte Größenverhältnisse) in der Anschauung darzulegen.' (Kant, 1922, p. 408)

[45]Recall that in the 'Analogies', cited above, Kant claims that quantitative relations expressed by proportions are *constitutive*, i.e. provide for the construction of an object. The *relation* is clearly not that which is under construction.

[46]In each passage of the *Critique* in which Kant provides an example of the construction of mathematical concepts, he refers to a figure, or geometric object. See passages at Bxii; A24; A48/B65; A164–5/B205–6; A220/B268; A223/B271; A240/B299; A713–4/B741–2; A716/B744; A718/B746; A722/B750; and A730–3/B758–61.

[47]Kant also refers to Segner's ostensive constructions of numbers in the *Prolegomena* (Kant, 1997, p. 269), as well as in a comment from 1790 (printed in the *Reflexionen zur Mathematik*) where he writes: 'Zahlbegriffe bedürfen eben so reinsinnlicher Bilder, e.g. Segner.' (Kant, 1911, p. 55)

appears in a footnote to a discussion of the Apollonian construction of a cone, is worth quoting in full:

> The following may serve to secure against misuse the expression 'construction of concepts' of which the *Critique of Pure Reason* speaks several times, and by means of which it for the first time has carefully distinguished between the procedure of reason in mathematics and philosophy. In the most general sense one can call construction all exhibition of a concept through the (spontaneous) production of a corresponding intuition. If it occurs through the mere imagination in accordance with an *a priori* concept, it is called pure construction. (These are the constructions which the mathematician must make use of in all his demonstrations.) Hence, he can demonstrate by means of a circle which he draws with his stick in the sand, no matter how irregular it may turn out to be, the attributes of a circle in general, as perfectly as if it had been etched on a copper plate by the greatest artist. If, however, it is practised on some kind of material it could be called empirical construction. The first can also be called *schematic*, the second *technical*. The latter, and really improperly named, construction (because it belongs not to science but to art and takes place by means of instruments) is either the *geometrical*, by means of compass and ruler, or the *mechanical*, for which other instruments are necessary as, for example, the drawing of the other conic sections besides the circle. (Allison, 1973, p. 111)

The foregoing passage makes clear that, insofar as Kant distinguishes between *kinds* of construction, the distinction is drawn between pure, or schematic construction and empirical, or technical construction. Consistent with Kant's own example (of the construction of the concept of a circle), both pure and empirical constructions are, in some sense, of geometrical figures. If the construction is pure, it is 'through the mere imagination in accordance with an *a priori* concept'; if empirical, it is 'practiced on some kind of material' and 'takes place by means of instruments'. Thus, the two sorts of construction are distinguished by the means whereby they are carried out: either the circle is 'drawn' by the imagination in accordance with an *a priori* concept of 'circle', or it is drawn with some empirical aids, like paper, pencil, and compass. Both of such constructions are ostensive in that both 'exhibit' a concept through the production of an intuition.[48]

The sort of construction that, for Kant, justifies a mathematical demonstration and distinguishes mathematical from philosophical reasoning is pure, schematic, *and* ostensive. In the particular case that algebra is applied to the solution of a geometric or arithmetic problem, such a construction might be *symbolized* for the sake of (algebraic) argument, rather than actually carried out. The possibility of

[48]Admittedly, Kant confuses the issue by further distinguishing *geometrical* empirical constructions from *mechanical* empirical constructions. His use of 'geometrical' here is misleading, since a pure schematic construction may be of a concept of geometry, for example, of a circle. Moreover, a 'geometrical' construction (by means of compass and ruler) can function as a pure intuition for the sake of mathematical, as opposed to mechanical, demonstration (Kant, 1998, A718/B746); Kant here follows Wolff, who distinguished 'Demonstratio mechanica, ein mechanischer Beweiß' from mathematical demonstration in his *Lexicon* (Wolff, 1965, pp. 506–7). An adequate interpretation of the difference between pure and empirical construction, and of why and how mathematical demonstration relies on the former, requires a closer look at Wolff's classification of demonstrations, as well as an interpretation of Kant's 'Schematism', both of which I will provide elsewhere.

its being carried out, by the imagination in accordance with *a priori* concepts and certain rules, is what allows such a 'symbolic construction' to stand in for its ostensive referent, the 'object itself'.

5.

My approach to understanding Kant's philosophy of mathematics in general, and his notion of 'symbolic construction' in particular, is based on the fact that Kant was deeply immersed in the textbook mathematics of the eighteenth century; only an understanding of these texts, on their own terms, can possibly provide the mathematical material and context required for a successful interpretation of Kant's philosophy of mathematics. Since Kant's philosophy of mathematics was developed relative to a specific body of mathematical practice quite distinct from that which currently obtains, our reading of Kant must not ignore the dissonance between the ontologies and methodologies of eighteenth- and twentieth-century mathematics. Moreover, by attending to the history of elementary mathematics we attain a deeper appreciation of the consistency of Kant's own views, so that his brief comments on algebra and 'symbolic construction' fit more naturally with, and even illuminate, his other stated views on the object and method of mathematics.

I have shown that in eighteenth-century elementary mathematics the individual algebraic symbols are not free variables ranging over an infinitely large set of natural numbers; rather they are conceived as symbols of the individual, ostensively constructible objects of arithmetic and geometry. Moreover, being a method of reasoning about geometrical and arithmetical problems, algebra does not have its own object independent of these constructible magnitudes. Consequently, in a Kantian context 'algebra' cannot be taken simply to denote the arithmetic of indeterminate or variable numeric quantities[49] but must be recognized as a method applied to the solution of arithmetic *and* geometric problems, resulting in a geometrical construction of 'magnitude in general': a line segment expressing either a number, or the determinate size of a magnitude (*quantum*).

The power of algebra in the eighteenth century came from its ability to isolate the component parts of a problem of geometry or arithmetic; for example, the triangle in the Hypotenuse Problem (Section 3) is seen as a whole, geometric object (a right triangle) made up of distinct, separable parts (legs, hypotenuse, altitude). Both the whole and the parts are conceived as magnitudes, or sized objects; the

[49]Gordon Brittan, like myself, has developed a position on 'symbolic construction' with respect to the same group of commentators, and has likewise rejected what he calls the 'calculational interpretation' of Kant's notion of symbolic construction (Brittan, 1992). For Brittan, such interpretations are distinguished by their assimilation of arithmetic to algebra, and their extension of Kant's symbolic construction to include arithmetical, as well as algebraic, reasoning. Though our initial approaches to the problem of analyzing this aspect of Kant's thought thus coincide, my suggestion for a new interpretation is entirely unlike that pursued by Brittan who claims that 'Kant is working his way to a rather modern and abstract conception of algebra as consisting of sets on which certain iterable operations are defined, a fact that demonstrates his originality and insight while it explains his obscurity.' (Brittan, 1992, p. 315) On the contrary, I have shown how Kant's view derives from his understanding of eighteenth-century algebraic practice, which is entirely unlike our 'modern and abstract conception'.

parts can thus be considered *as* parts of the whole object, or separately as objects, or magnitudes, in themselves. The application of algebra to such a problem isolates the individual components of the problem as magnitudes by assigning each a symbol; in this way the individual components of the problem are mirrored by the algebraic symbolism. For the algebraist, this 'symbolic construction', that is, the symbolic representation of a geometrically constructible object, is a heuristic aid to elementary problem solving: the symbolic manipulation of magnitudes and relations among them has the advantage of being more clear and less cumbersome than purely geometric construction.[50]

Kant explains both the difference between the mathematical and philosophical methods and the syntheticity of mathematical judgments by virtue of the fact that mathematical concepts are constructed in intuition (Kant, 1998, A713/B741). His examples of such constructions, and indeed of mathematical knowledge in general, rely on the paradigm of Euclidean geometry and its postulates for constructing geometric figures; even arithmetic cognition relies on the construction of strokes or points. His mention of algebra and its 'symbolic construction' serves to extend his theory of mathematical cognition to include the so-called 'analytic arts' of eighteenth century mathematical practice, and thereby show how the algebraic method likewise yields synthetic judgments. By symbolizing the performable constructions of algebraic concepts such as, for example, 'mere magnitude', the algebraist derives equations that express synthetic judgments as well as the ostensive constructions they represent. So, the derivation of Equation * 'goes outside' the concept of the hypotenuse of the triangle, symbolized by x, to show that it is equivalent to the concept of half the perimeter of the triangle diminished by twice the area divided by the perimeter. Ultimately, such a synthetic judgment as that expressed symbolically by Equation * is justified by the geometrical construction of the root of the equation, x.

I hope to have demonstrated the power of a contextualist approach that sees Kant as situated in the mathematics of his time and seeks to understand his philosophy of mathematics in relation to the mathematical practice with which he was engaged. Elementary mathematics as Kant conceived it relies fundamentally on the paradigmatically ostensive constructions of plane geometry: these 'constructions of mathematical concepts' justify the synthetic *a priori* judgments of mathematics and distinguish mathematical from philosophical cognition. My reading of Kant's theory of algebraic cognition and 'symbolic construction' has the advantage of showing how these two brief passages do not defy but rather support both Kant's general conclusions about the nature of mathematics and, insofar as the possibility of mathematical knowledge plays a role in justifying transcendental idealism, the critical philosophy as a whole.

[50]Of course, in a post-Kantian context, the algebraic equation itself becomes an object of mathematical investigation; at this later stage, algebraic notations have the capacity to function not merely heuristically but also constitutively. This is one reason why the assimilation of algebra to arithmetic seems at first quite natural. I am grateful to Madeline Muntersbjorn for bringing this point to my attention.

Acknowledgements—I would like to thank Gary Hatfield, Paul Guyer, Scott Weinstein, Madeline Muntersjborn, Lanier Anderson and two anonymous referees for their helpful comments and suggestions.

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Appendix

Here I will show how Wolff derives Equation * for the hypotenuse of the triangle in the Hypotenuse Problem discussed in Section 3, above (Wolff, 1968, p. 388). First, Wolff lets

$$\mathrm{AB} + \mathrm{BC} + \mathrm{CA} = a;$$

$$\mathrm{AC} = x;$$

$$\text{and, the area of } \triangle\mathrm{ABC} = b^2.$$

It follows from the given conditions and the relations made evident in the initial schematic figure (Fig. 2), that the sum of the legs is equal to the perimeter diminished by the length of the hypotenuse:

$$\mathrm{AB} + \mathrm{BC} = a - x; \tag{1}$$

that, by the Pythagorean theorem:

$$\mathrm{AC}^2 = \mathrm{AB}^2 + \mathrm{BC}^2; \tag{2}$$

and, that the area of the right triangle is equal to 1/2 the product of the legs:[51]

$$1/2(\mathrm{AB}\cdot\mathrm{BC}) = b^2. \tag{3}$$

So, by applying standard theorems of elementary geometry, Wolff has found three algebraic equations expressing relations among known and unknown elements of the problem. He then algebraically manipulates Equation (2), showing that

$$\mathrm{AC}^2 = \mathrm{AB}^2 + \mathrm{BC}^2 = (\mathrm{AB} + \mathrm{BC})^2 - 2(\mathrm{AB}\cdot\mathrm{BC}). \tag{2$'$}$$

Now, wanting to transform Equation (2′) into a quadratic equation in x using the known magnitudes a and b^2, Wolff makes the following observations: since it is given that AC $= x$, so

$$\mathrm{AC}^2 = x^2;$$

likewise, squaring both sides of Equation (1),

$$(\mathrm{AB} + \mathrm{BC})^2 = (a - x)^2; \tag{1$'$}$$

finally, by transformation of Equation (3),

$$2(\mathrm{AB}\cdot\mathrm{BC}) = 4b^2. \tag{3$'$}$$

By substitution and transformation of Equation (2′), we have

$$x^2 = (a - x)^2 - 4b^2$$

$$x^2 = a^2 - 2ax + x^2 - 4b^2$$

$$2ax = a^2 - 4b^2$$

$$x = 1/2\cdot a - 2b^2/a. \tag{A*}$$

Wolff now has an expression for the unknown magnitude x, the length of the hypotenuse AC, in terms of the known magnitudes a and b^2.

[51]Since Wolff has not designated the legs by separate letters, he here continues to use their geometric designations (e.g. AB, AC) in the equations expressing their relationships.